Graphing Rational Functions

24 problems with a three problem challenge set. For every problem give the excluded values, any holes with their coordinates, the vertical asymptotes, all x-intercepts and the y-intercept if they exist, and the end behavior. If a feature does not exist, write “none.” Parts C and D ask for one further step, named in the heading of that part. Blank grids for sketching start on page 6, and the answer key with completed graphs follows them.

The six steps

1. Factor the numerator and the denominator completely.
2. Excluded values come from the **original** denominator.
3. Cancel, then read the **reduced** denominator. Its zeros are the vertical asymptotes. Any other excluded value is a hole.
4. Intercepts: reduced numerator = 0 for the x-intercepts, discarding excluded values. Evaluate at x = 0 for the y-intercept.
5. End behavior by comparing degrees in the reduced form.
6. Sign chart: cut at every real zero of the reduced numerator and the reduced denominator, then test one allowed value per interval.

Part A. Read the features from a factored function

Everything is factored already. The work is deciding what each factor does.

1. $f(x) = \dfrac{x + 2}{x - 3}$

Excluded values _____
Holes _____
Vertical asymptotes _____
x-intercepts _____
y-intercept _____
End behavior _____

2. $f(x) = \dfrac{x - 4}{(x + 1)(x - 5)}$

Excluded values _____
Holes _____
Vertical asymptotes _____
x-intercepts _____
y-intercept _____
End behavior _____

3. $f(x) = \dfrac{3x}{x + 2}$

Excluded values _____
Holes _____
Vertical asymptotes _____
x-intercepts _____
y-intercept _____
End behavior _____

4. $f(x) = \dfrac{(x - 1)(x + 4)}{(x - 1)(x + 2)}$

Excluded values _____
Holes _____
Vertical asymptotes _____
x-intercepts _____
y-intercept _____
End behavior _____

5. $f(x) = \dfrac{5}{(x - 2)^2}$

Excluded values _____
Holes _____
Vertical asymptotes _____
x-intercepts _____
y-intercept _____
End behavior _____

6. $f(x) = \frac{x^2 - 9}{x + 3}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Part B. Factor first, then find every feature

Two of these six have a factor that cancels. One of those two still has an asymptote where it cancelled.

7. $f(x) = \frac{x^2 - 4}{x^2 - x - 6}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

8. $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

9. $f(x) = \frac{2x^2 - 8}{x^2 - 5x + 6}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

10. $f(x) = \frac{x^2 - 1}{x^2 + 2x + 1}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

11. $f(x) = \frac{x^3 - 4x}{x^2 - 2x}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

12. $f(x) = \frac{x^2 - 16}{x^2 - 8x + 16}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Part C. Sign charts and branch direction

Build the sign chart, then say which side of the x-axis each piece of the graph sits on.

13.

$$f(x) = \frac{x - 1}{(x + 2)(x - 3)}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Interval	Test value	Sign

Cut points _____

Which side of the x-axis does each branch sit on?
_____

14.

$$f(x) = \frac{x^2}{x - 1}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Interval	Test value	Sign

Cut points _____

Which side of the x-axis does each branch sit on?
_____

15.

$$f(x) = \frac{(x + 1)^2}{(x - 2)(x + 3)}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Interval	Test value	Sign

Cut points _____

Which side of the x-axis does each branch sit on?
_____

16.

$$f(x) = \frac{x^2 - 1}{x}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Interval	Test value	Sign

Cut points _____

Which side of the x-axis does each branch sit on?
_____

17.

$$f(x) = \frac{x^2}{x}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Interval	Test value	Sign

Cut points _____

Which side of the x-axis does each branch sit on?
_____

18.

$$f(x) = \frac{4 - x^2}{x^2 - 1}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Interval	Test value	Sign

Cut points _____

Which side of the x-axis does each branch sit on?
_____

Part D. End behavior, slant asymptotes, and meeting the asymptote

For each function, state its end behavior. If it has a horizontal or slant asymptote, solve for any points where the graph meets that line.

19.

$$f(x) = \frac{x^2 + 1}{x - 2}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Where the graph meets that line _____

20.

$$f(x) = \frac{3x^2 - 2x}{x^2 + 1}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Where the graph meets that line _____

21.

$$f(x) = \frac{x^2 - 4x + 3}{x - 1}$$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Where the graph meets that line _____

22. $f(x) = \dfrac{2x}{x^2 - 9}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Where the graph meets that line _____

23. $f(x) = \dfrac{x^3}{x^2 - 1}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Where the graph meets that line _____

24. $f(x) = \dfrac{x^2 + 2x + 1}{x^2 + 1}$

Excluded values
Holes
Vertical asymptotes
x-intercepts
y-intercept
End behavior

Where the graph meets that line _____

Challenge set

Three problems that test the rules rather than the procedure.

C1. Find every point where the graph of $f(x) = (x^4 + x^2 - 2x + 2) / (x^4 + 1)$ meets its horizontal asymptote, and say whether it crosses or touches at each one.

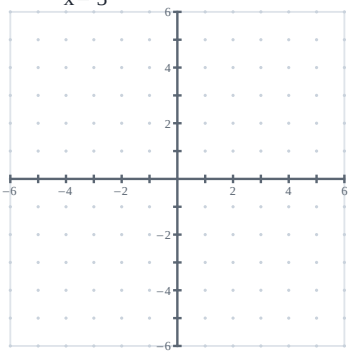
C2. The graph of $f(x) = x^3 / (x^4 + 1)$ meets its horizontal asymptote at a root of multiplicity 3. Does it cross or touch there? Explain which property of the multiplicity decides it.

C3. Write a rational function with a vertical asymptote at $x = -2$, a hole at $x = 3$, an x-intercept at $x = 1$, and a horizontal asymptote $y = 2$. Then write a second function, built from different factors, that has the same four features.

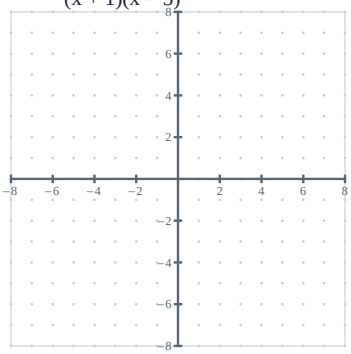
Sketch the graphs

One grid for each of the 24 problems, in the same order. Mark the asymptotes as dashed lines first, then the intercepts and any holes, then draw each branch using your sign chart. The grids are square except where a slant asymptote needs a taller y range.

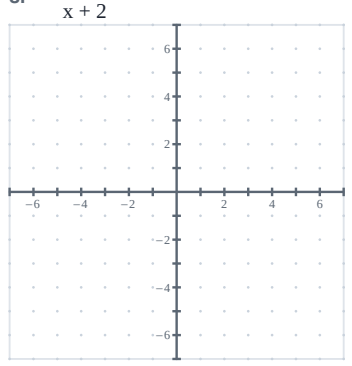
1. $\dfrac{x+2}{x-3}$



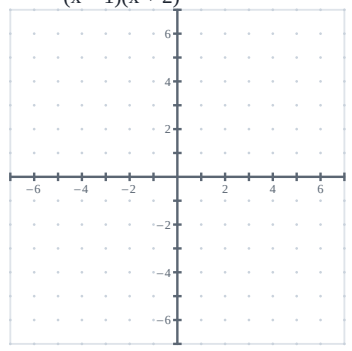
2. $\dfrac{x-4}{(x+1)(x-5)}$



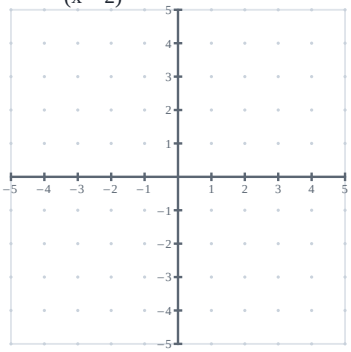
3. $\dfrac{3x}{x+2}$



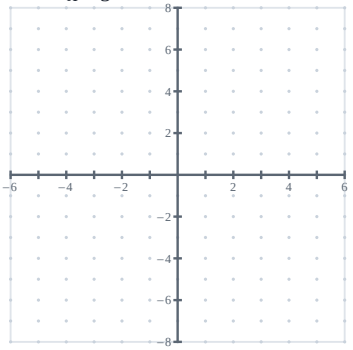
4. $\dfrac{(x-1)(x+4)}{(x-1)(x+2)}$



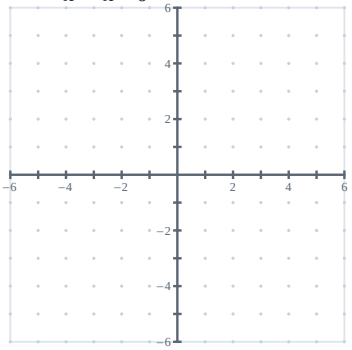
5. $\dfrac{5}{(x-2)^2}$



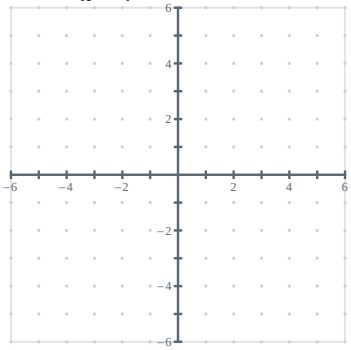
6. $\dfrac{x^2-9}{x+3}$



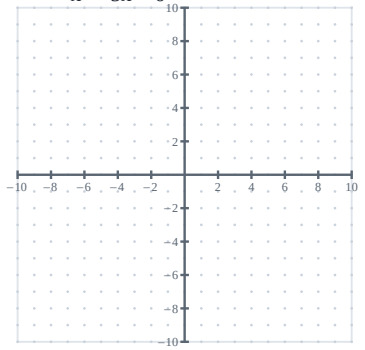
7. $\dfrac{x^2-4}{x^2-x-6}$



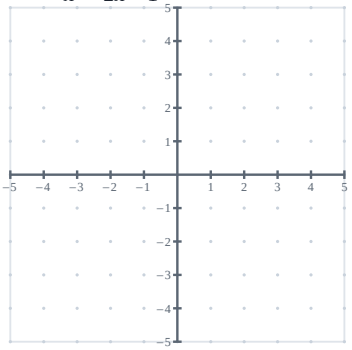
8. $\dfrac{x^2+x-6}{x^2-4}$



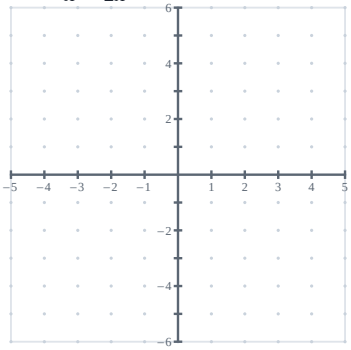
9. $\dfrac{2x^2-8}{x^2-5x+6}$



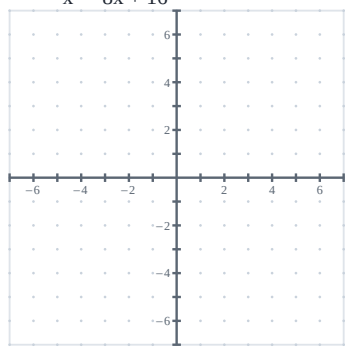
10. $\dfrac{x^2-1}{x^2+2x+1}$



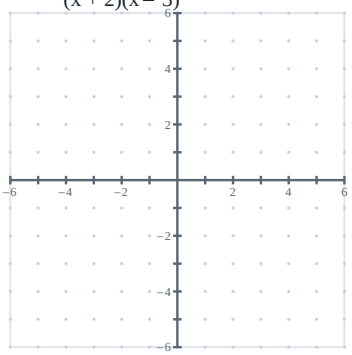
11. $\dfrac{x^3-4x}{x^2-2x}$



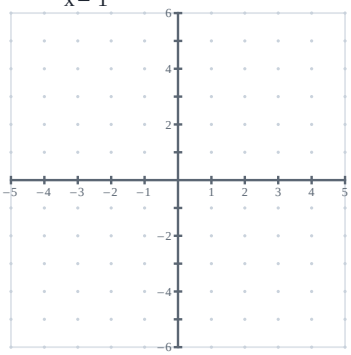
12. $\dfrac{x^2-16}{x^2-8x+16}$



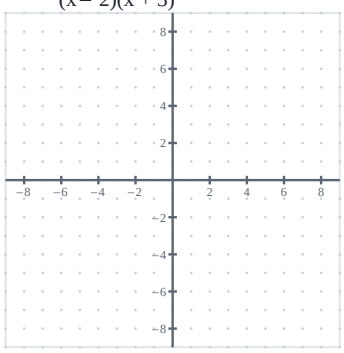
13. $\dfrac{x-1}{(x+2)(x-3)}$



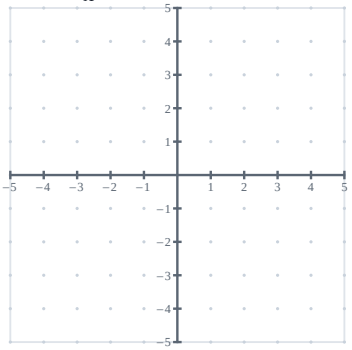
14. $\dfrac{x^2}{x-1}$



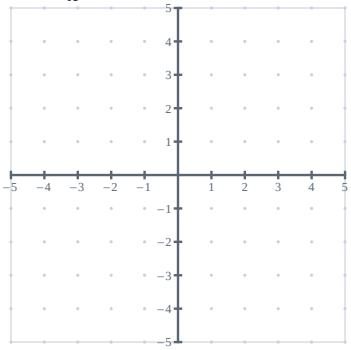
15. $\dfrac{(x+1)^2}{(x-2)(x+3)}$



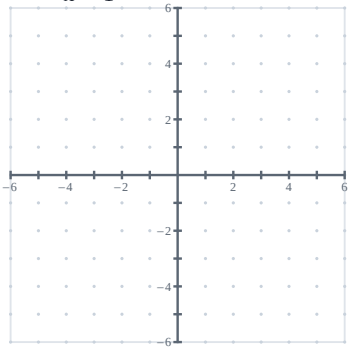
16. $\dfrac{x^2-1}{x}$



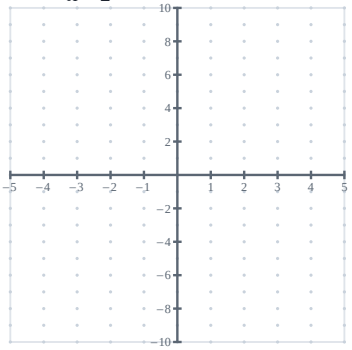
17. $\dfrac{x^2}{x}$



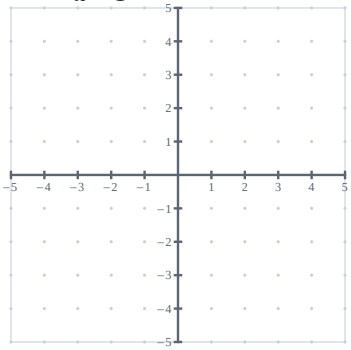
18. $\dfrac{4-x^2}{x^2-1}$



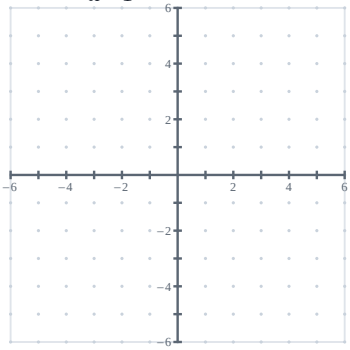
19. $\dfrac{x^2+1}{x-2}$



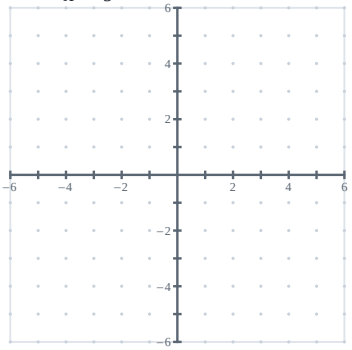
20. $\dfrac{3x^2-2x}{x^2+1}$



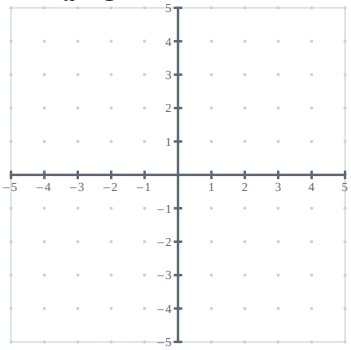
21. $\dfrac{x^2-4x+3}{x-1}$



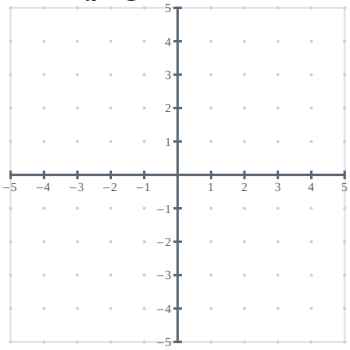
22. $\dfrac{2x}{x^2-9}$



23. $\dfrac{x^3}{x^2-1}$



24. $\dfrac{x^2+2x+1}{x^2+1}$



Answer key

Every feature for all 24 problems, then the completed graphs in the same order. Sign charts are listed for Part C, where they are the point of the exercise.

1. $\frac{x+2}{x-3}$

Excluded	3
Holes	none
Vert. asymptotes	$x = 3$
x-intercepts	$(-2, 0)$
y-intercept	$(0, -2/3)$
End behavior	horizontal asymptote $y = 1$
Meets that line	never meets it

2. $\frac{x-4}{(x+1)(x-5)}$

Excluded	$-1, 5$
Holes	none
Vert. asymptotes	$x = -1$, $x = 5$
x-intercepts	$(4, 0)$
y-intercept	$(0, 4/5)$
End behavior	horizontal asymptote $y = 0$
Meets that line	crosses at $(4, 0)$

3. $\frac{3x}{x+2}$

Excluded	-2
Holes	none
Vert. asymptotes	$x = -2$
x-intercepts	$(0, 0)$
y-intercept	$(0, 0)$
End behavior	horizontal asymptote $y = 3$
Meets that line	never meets it

4. $\frac{(x-1)(x+4)}{(x-1)(x+2)}$

Excluded	$-2, 1$
Holes	$(1, 5/3)$
Vert. asymptotes	$x = -2$
x-intercepts	$(-4, 0)$
y-intercept	$(0, 2)$
End behavior	horizontal asymptote $y = 1$
Meets that line	never meets it

5. $\frac{5}{(x-2)^2}$

Excluded	2
Holes	none
Vert. asymptotes	$x = 2$
x-intercepts	none
y-intercept	$(0, 5/4)$
End behavior	horizontal asymptote $y = 0$
Meets that line	never meets it

6. $\frac{x^2-9}{x+3}$

Excluded	-3
Holes	$(-3, -6)$
Vert. asymptotes	none
x-intercepts	$(3, 0)$
y-intercept	$(0, -3)$
End behavior	no asymptote, the graph is a polynomial with a hole

7. $\frac{x^2-4}{x^2-x-6}$

Excluded	$-2, 3$
Holes	$(-2, 4/5)$
Vert. asymptotes	$x = 3$
x-intercepts	$(2, 0)$
y-intercept	$(0, 2/3)$
End behavior	horizontal asymptote $y = 1$
Meets that line	never meets it

8. $\frac{x^2+x-6}{x^2-4}$

Excluded	$-2, 2$
Holes	$(2, 5/4)$
Vert. asymptotes	$x = -2$
x-intercepts	$(-3, 0)$
y-intercept	$(0, 3/2)$
End behavior	horizontal asymptote $y = 1$
Meets that line	never meets it

9. $\frac{2x^2-8}{x^2-5x+6}$

Excluded	2, 3
Holes	$(2, -8)$
Vert. asymptotes	$x = 3$
x-intercepts	$(-2, 0)$
y-intercept	$(0, -4/3)$
End behavior	horizontal asymptote $y = 2$
Meets that line	never meets it

10. $\frac{x^2-1}{x^2+2x+1}$

Excluded	-1
Holes	none
Vert. asymptotes	$x = -1$
x-intercepts	$(1, 0)$
y-intercept	$(0, -1)$
End behavior	horizontal asymptote $y = 1$
Meets that line	never meets it

11. $\frac{x^3 - 4x}{x^2 - 2x}$

Excluded	0, 2
Holes	(0, 2), (2, 4)
Vert. asymptotes	none
x-intercepts	(−2, 0)
y-intercept	none, x = 0 is excluded
End behavior	no asymptote, the graph is a polynomial with a hole

12. $\frac{x^2 - 16}{x^2 - 8x + 16}$

Excluded	4
Holes	none
Vert. asymptotes	x = 4
x-intercepts	(−4, 0)
y-intercept	(0, −1)
End behavior	horizontal asymptote y = 1
Meets that line	never meets it

13. $\frac{x - 1}{(x + 2)(x - 3)}$

Excluded	−2, 3
Holes	none
Vert. asymptotes	x = −2, x = 3
x-intercepts	(1, 0)
y-intercept	(0, 1/6)
End behavior	horizontal asymptote y = 0
Meets that line	crosses at (1, 0)
Cut points	−2, 1, 3
Signs	(−∞, −2) − (−2, 1) + (1, 3) − (3, ∞) +

14. $\frac{x^2}{x - 1}$

Excluded	1
Holes	none
Vert. asymptotes	x = 1
x-intercepts	(0, 0) mult 2
y-intercept	(0, 0)
End behavior	slant asymptote y = x + 1
Meets that line	never meets it
Cut points	0, 1
Signs	(−∞, 0) − (0, 1) − (1, ∞) +

15. $\frac{(x + 1)^2}{(x - 2)(x + 3)}$

Excluded	−3, 2
Holes	none
Vert. asymptotes	x = −3, x = 2
x-intercepts	(−1, 0) mult 2
y-intercept	(0, −1/6)
End behavior	horizontal asymptote y = 1
Meets that line	crosses at (−7, 1)
Cut points	−3, −1, 2
Signs	(−∞, −3) + (−3, −1) − (−1, 2) − (2, ∞) +

16. $\frac{x^2 - 1}{x}$

Excluded	0
Holes	none
Vert. asymptotes	x = 0
x-intercepts	(−1, 0), (1, 0)
y-intercept	none, x = 0 is excluded
End behavior	slant asymptote y = x
Meets that line	never meets it
Cut points	−1, 0, 1
Signs	(−∞, −1) − (−1, 0) + (0, 1) − (1, ∞) +

17. $\frac{x^2}{x}$

Excluded	0
Holes	(0, 0)
Vert. asymptotes	none
x-intercepts	none
y-intercept	none, x = 0 is excluded
End behavior	no asymptote, the graph is a polynomial with a hole
Cut points	0
Signs	(−∞, 0) − (0, ∞) +

18. $\frac{4 - x^2}{x^2 - 1}$

Excluded	−1, 1
Holes	none
Vert. asymptotes	x = −1, x = 1
x-intercepts	(−2, 0), (2, 0)
y-intercept	(0, −4)
End behavior	horizontal asymptote y = −1
Meets that line	never meets it
Cut points	−2, −1, 1, 2
Signs	(−∞, −2) − (−2, −1) + (−1, 1) − (1, 2) + (2, ∞) −

19. $\frac{x^2 + 1}{x - 2}$

Excluded	2
Holes	none
Vert. asymptotes	x = 2
x-intercepts	none
y-intercept	(0, −1/2)
End behavior	slant asymptote y = x + 2
Meets that line	never meets it

20. $\frac{3x^2 - 2x}{x^2 + 1}$

Excluded	none
Holes	none
Vert. asymptotes	none
x-intercepts	(0, 0), (2/3, 0)
y-intercept	(0, 0)
End behavior	horizontal asymptote y = 3
Meets that line	crosses at (−3/2, 3)

21. $\frac{x^2 - 4x + 3}{x - 1}$

Excluded	1
Holes	$(1, -2)$
Vert. asymptotes	none
x-intercepts	(3, 0)
y-intercept	$(0, -3)$
End behavior	no asymptote, the graph is a polynomial with a hole

22. $\frac{2x}{x^2 - 9}$

Excluded	$-3, 3$
Holes	none
Vert. asymptotes	$x = -3$, $x = 3$
x-intercepts	(0, 0)
y-intercept	(0, 0)
End behavior	horizontal asymptote $y = 0$
Meets that line	crosses at (0, 0)

23. $\frac{x^3}{x^2 - 1}$

Excluded	$-1, 1$
Holes	none
Vert. asymptotes	$x = -1$, $x = 1$
x-intercepts	(0, 0) mult 3
y-intercept	(0, 0)
End behavior	slant asymptote $y = x$
Meets that line	crosses at (0, 0)

24. $\frac{x^2 + 2x + 1}{x^2 + 1}$

Excluded	none
Holes	none
Vert. asymptotes	none
x-intercepts	$(-1, 0)$ mult 2
y-intercept	(0, 1)
End behavior	horizontal asymptote $y = 1$
Meets that line	crosses at (0, 1)

Challenge set

C1. The denominator $x^4 + 1$ is never zero, so nothing is excluded. Equal degrees with leading coefficients 1 and 1, so the horizontal asymptote is $y = 1$. Solve $(x^4 + x^2 - 2x + 2) - 1(x^4 + 1) = 0$, which reduces to $x^2 - 2x + 1 = 0$, that is $(x - 1)^2 = 0$. The only solution is $x = 1$, with multiplicity 2. Even multiplicity, so the graph **touches y = 1 at (1, 1)** and stays above it on both sides. Touching an asymptote is not limited to the x-axis.

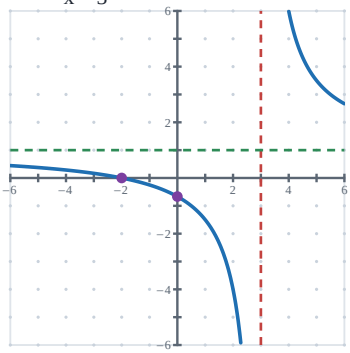
C2. It **crosses**. Solving $f(x) = 0$ gives $x^3 = 0$, a root of multiplicity 3, and 3 is odd. The test is odd against even multiplicity, not single against repeated. An odd multiplicity means $f(x) - k$ changes sign as x passes through the root, which is what crossing means.

C3. Many functions work. One answer is **$f(x) = 2(x - 1)(x - 3) / ((x + 2)(x - 3))$**. The factor $(x - 3)$ cancels completely, so $x = 3$ is a hole, and the reduced function $2(x - 1)/(x + 2)$ gives the height 4/5, putting the hole at (3, 4/5). The vertical asymptote is $x = -2$, the x-intercept is $x = 1$, and the reduced degrees are 1 and 1 with leading coefficients 2 and 1, so the horizontal asymptote is $y = 2$. A second answer built from different factors is **$f(x) = 2(x - 1)(x - 3)(x^2 + 1) / ((x + 2)(x - 3)(x^2 + 2))$**, where the hole is at (3, 8/11) and every other feature is unchanged. The denominator's copies of $(x - 3)$ have to cancel completely, so the numerator needs at least as many, and extra copies in the numerator are allowed. What is fixed is the ratio of leading coefficients after cancelling, which the horizontal asymptote pins at 2.

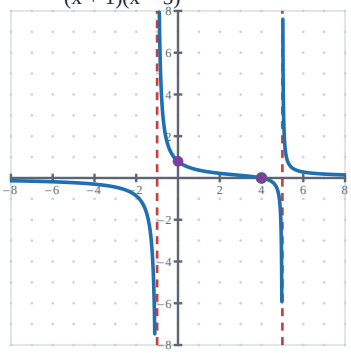
Answer key: completed graphs

Blue is the function. Red dashed lines are vertical asymptotes and red open circles are holes. Green dashed lines are the horizontal or slant asymptote, and a green dot marks a point where the graph meets it. Purple dots are the intercepts. When the horizontal asymptote is y = 0 it is the x-axis itself, so no separate green line is drawn, and the point where the graph meets it is also an x-intercept.

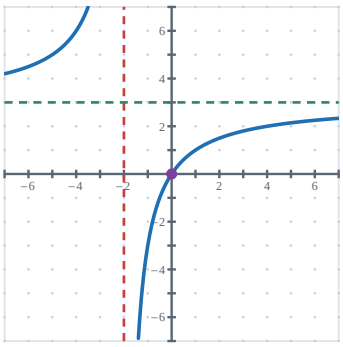
1. $\frac{x+2}{x-3}$



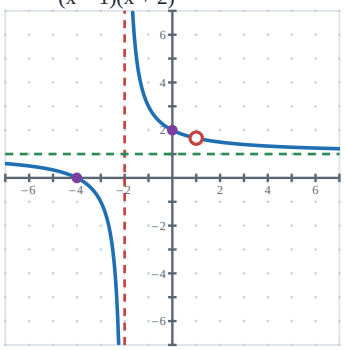
2. $\frac{x-4}{(x+1)(x-5)}$



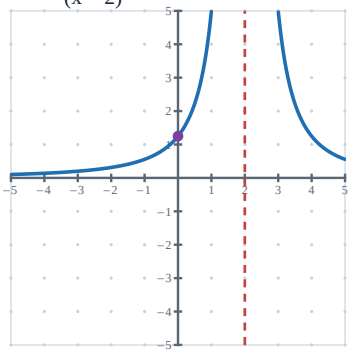
3. $\frac{3x}{x+2}$



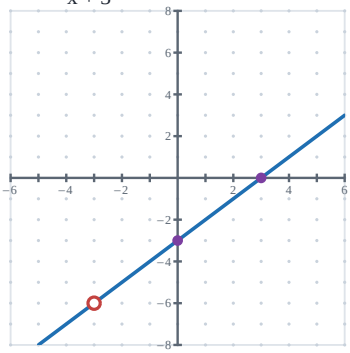
4. $\frac{(x-1)(x+4)}{(x-1)(x+2)}$



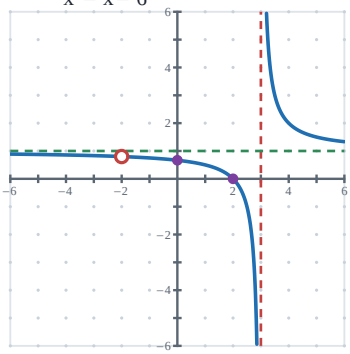
5. $\frac{5}{(x-2)^2}$



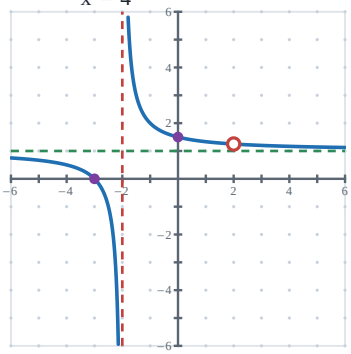
6. $\frac{x^2-9}{x+3}$



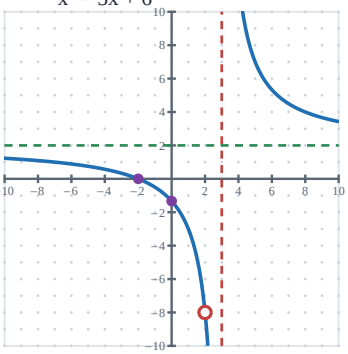
7. $\frac{x^2-4}{x^2-x-6}$



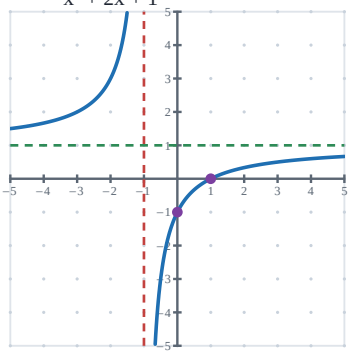
8. $\frac{x^2+x-6}{x^2-4}$



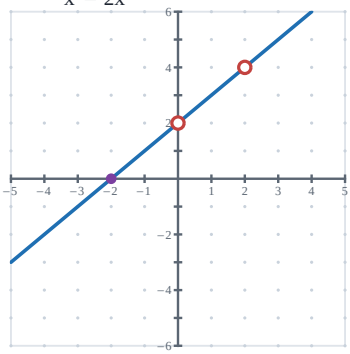
9. $\frac{2x^2-8}{x^2-5x+6}$



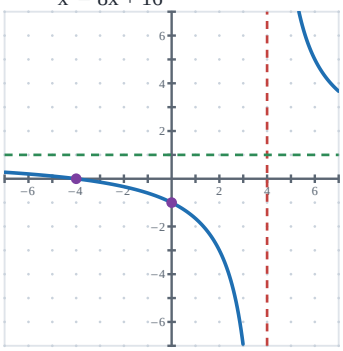
10. $\frac{x^2-1}{x^2+2x+1}$



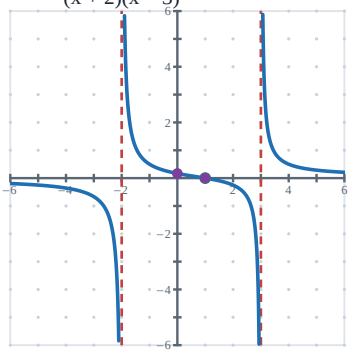
11. $\frac{x^3-4x}{x^2-2x}$



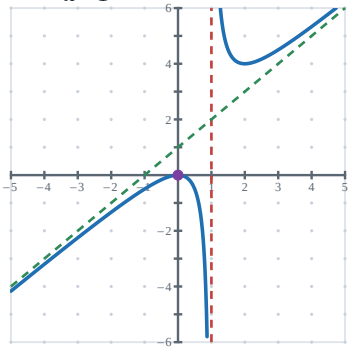
12. $\frac{x^2-16}{x^2-8x+16}$



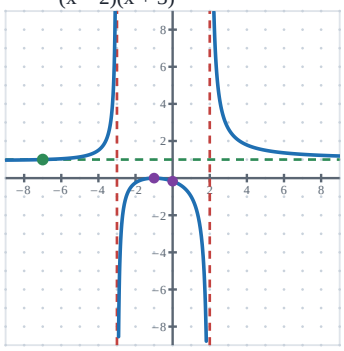
13. $\dfrac{x-1}{(x+2)(x-3)}$



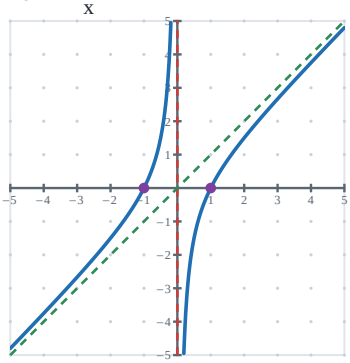
14. $\dfrac{x^2}{x-1}$



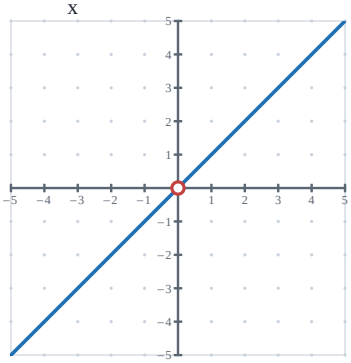
15. $\dfrac{(x+1)^2}{(x-2)(x+3)}$



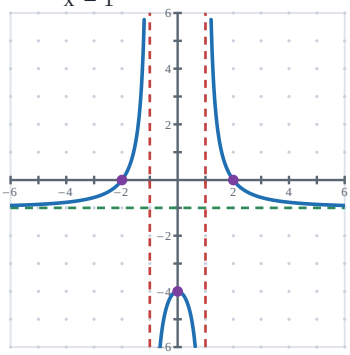
16. $\dfrac{x^2-1}{x}$



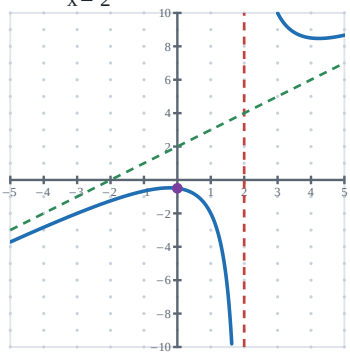
17. $\dfrac{x^2}{x}$



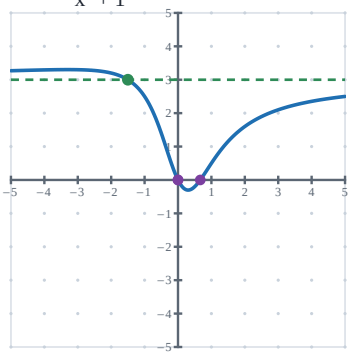
18. $\dfrac{4-x^2}{x^2-1}$



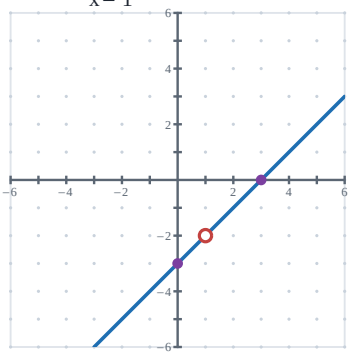
19. $\dfrac{x^2+1}{x-2}$



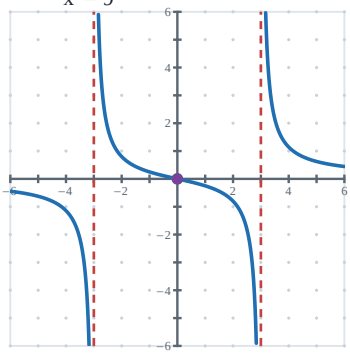
20. $\dfrac{3x^2-2x}{x^2+1}$



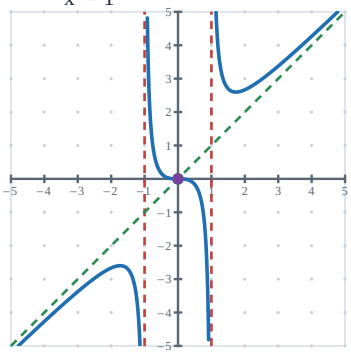
21. $\dfrac{x^2-4x+3}{x-1}$



22. $\dfrac{2x}{x^2-9}$



23. $\dfrac{x^3}{x^2-1}$



24. $\dfrac{x^2+2x+1}{x^2+1}$

