Complex Rational Expressions

Simplifying complex fractions, with every excluded value. 27 problems and a complete answer key.

Name _____ Date _____

METHOD A: COMBINE, THEN DIVIDE

1. Write the top level as one fraction.
2. Write the bottom level as one fraction.
3. Multiply the top by the reciprocal of the bottom.
4. Factor everything, then cancel common factors.

METHOD B: MULTIPLY BY THE LCD

1. Factor every inner denominator.
2. Build the LCD of all inner fractions, both levels.
3. Multiply every term on both levels by that LCD.
4. Factor the result, then cancel common factors.

THE THREE RESTRICTION SOURCES

- **1.** Every inner denominator in the top level is nonzero.
- **2.** Every inner denominator in the bottom level is nonzero.
- **3.** The whole bottom level is nonzero. Combine or clear the bottom, set the result equal to zero, and keep the solutions that the inner denominators allow.

RULES THAT DECIDE THE ANSWER

- Start the list with the inner-denominator restrictions, then add the lower-expression ones as soon as you combine or clear the bottom.
- Cancel factors only. Never cancel across a plus or minus sign.
- Rewrite every negative exponent as a fraction first.
- When an entire expression is raised to a negative power, its base becomes a denominator and must be nonzero.

WORKED IN ONE LINE

$$\frac{\frac{1}{x}}{\frac{1}{x}-\frac{1}{2}} = \frac{2}{2-x} \qquad x \neq 0, 2$$

Sources 1 and 2 give $x \neq 0$ from the inner denominator. Source 3 gives $x \neq 2$, because the bottom level is zero there. The simplified form shows only $x \neq 2$, so the other restriction has to be carried down from the first line.

How to use this worksheet

Work each problem on the dotted lines, or on your own paper if you need more room. Every answer has two parts: the simplified expression and the complete list of excluded values. Start the list with the inner-denominator restrictions, add the lower-expression restrictions as soon as you combine or clear the bottom, and keep the full list through simplification. The answer key begins after the challenge problems.

Which method to use

	Method A: combine, then divide	Method B: multiply by the LCD
Usually convenient when	Each level is already one fraction, or can be combined quickly.	Several fractional terms share a manageable LCD.
Finding Source 3	Set the numerator of the combined bottom to zero, and keep the solutions the inner denominators allow.	Set the cleared bottom to zero, then discard any solution that is already an excluded value.
Main risk	Flipping the wrong fraction.	Multiplying only some of the terms.

Adapted from problem types I use in Algebra II and Precalculus tutoring sessions. Free to print and to use in class.

Part A · Reading the main fraction bar as division

Mostly numeric warm ups. Where a level is a sum, combine it into a single fraction first, then divide.

1. $\dfrac{\frac{2}{3}}{\frac{4}{9}}$

2. $\dfrac{\frac{5}{6}}{\frac{10}{3}}$

3. $\dfrac{1+\frac{1}{2}}{1-\frac{1}{4}}$

4. $\dfrac{\frac{2}{x}}{\frac{6}{x^2}}$

5. $\dfrac{\frac{x}{4}}{\frac{x^2}{8}}$

6. $\dfrac{\frac{x+1}{x-3}}{\frac{x+1}{x+3}}$

Part B · Combine each level, then divide

Method A. Watch for the restriction that comes from the combined bottom.

7. $$\frac{\frac{1}{x}+\frac{1}{4}}{\frac{1}{x}-\frac{1}{4}}$$

8. $$\frac{\frac{1}{2}-\frac{1}{x}}{\frac{1}{2}+\frac{1}{x}}$$

9. $$\frac{3+\frac{6}{x}}{1-\frac{4}{x^2}}$$

10. $$\frac{1-\frac{9}{x^2}}{1+\frac{3}{x}}$$

11. $$\frac{\frac{x}{x+2}+1}{\frac{x}{x+2}-1}$$

12. $$\frac{\frac{2}{x-1}-2}{\frac{2}{x-1}+2}$$

Part C · Multiply both levels by the LCD

Method B. Factor every inner denominator before building the LCD.

13. $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$

14. $\dfrac{\frac{1}{x+1}+\frac{1}{x-1}}{\frac{1}{x+1}-\frac{1}{x-1}}$

15. $\dfrac{\frac{3}{x-2}}{\frac{1}{x-2}+\frac{1}{x+2}}$

16. $\dfrac{\frac{1}{x^2-9}}{\frac{1}{x-3}+\frac{1}{x+3}}$

17. $\dfrac{\frac{5}{x}-\frac{5}{x+1}}{\frac{1}{x^2+x}}$

18. $\dfrac{\frac{1}{x-4}-\frac{1}{x}}{\frac{1}{x-4}+\frac{1}{x}}$

Part D · Mixed, with negative exponents and factoring

Choose your own method. Factor first where a denominator is quadratic.

19. $\dfrac{x^{-1} + 2}{x^{-1} - 2}$

20. $(a^{-1} + b^{-1})^{-1}$

21. $\dfrac{\dfrac{1}{x^2 - x - 6}}{\dfrac{1}{x - 3}}$

22. $\dfrac{\dfrac{x + 3}{x^2 - 4}}{\dfrac{x^2 + 6x + 9}{x + 2}}$

23. $\dfrac{4 - \dfrac{1}{x^2}}{2 + \dfrac{1}{x}}$

24. $\dfrac{1 + \dfrac{2}{x} - \dfrac{15}{x^2}}{1 + \dfrac{7}{x} + \dfrac{10}{x^2}}$

Challenge problems

Problem 25 is the difference quotient from calculus. Problem 26 has three levels. Problem 27 asks you to build the expression first.

25.

$$\frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$$

26.

$$\frac{1}{1 + \dfrac{1}{1 + \dfrac{1}{x}}}$$

27. A tutor drives from Fremont to a lesson in Union City at an average speed of r_1 miles per hour and returns along the same route at an average speed of r_2 miles per hour. Write the average speed for the whole round trip as a single simplified rational expression in r_1 and r_2. State both the algebraic exclusions and the conditions the situation itself imposes on the two speeds. Then compute the average speed for a route driven out at 20 mph and back at 45 mph, and explain why the answer is not 32.5 mph.

Answer key

Each entry gives the simplified expression and the complete list of excluded values. A restriction still counts even when it is invisible in the simplified form.

1. $\frac{3}{2}$

2. $\frac{1}{4}$

3. 2

4. $\frac{x}{3}$

$x \neq 0$

5. $\frac{2}{x}$

$x \neq 0$

The only restriction comes from the main denominator.

6. $\frac{x+3}{x-3}$

$x \neq -3, -1, 3$

The simplified form shows only $x \neq 3$; the other two must be carried down.

7. $\frac{x+4}{4-x}$

$x \neq 0, 4$

8. $\frac{x-2}{x+2}$

$x \neq -2, 0$

9. $\frac{3x}{x-2}$

$x \neq -2, 0, 2$

The combined bottom is $(x^2 - 4)/x^2$, so Source 3 gives both 2 and −2.

10. $\frac{x-3}{x}$

$x \neq -3, 0$

11. $-(x+1)$

$x \neq -2$

The combined bottom is $-2/(x + 2)$, whose numerator is never zero, so there is no Source 3 restriction.

12. $\frac{2-x}{x}$

$x \neq 0, 1$

13. $\frac{x+y}{y-x}$

$x \neq 0, y \neq 0, x \neq y$

14. $-x$

$x \neq -1, 1$

15. $\frac{3x+6}{2x}$

$x \neq -2, 0, 2$

16. $\frac{1}{2x}$

$x \neq -3, 0, 3$

17. 5

$x \neq -1, 0$

A constant result still carries the restrictions of the original expression.

18. $\frac{2}{x-2}$

$x \neq 0, 2, 4$

19. $\frac{2x+1}{1-2x}$

$x \neq 0, 1/2$

20. $\frac{ab}{a+b}$

$a \neq 0, b \neq 0, a + b \neq 0$

21. $\dfrac{1}{x+2}$

$x \neq -2, 3$

22. $\dfrac{1}{(x-2)(x+3)}$

$x \neq -3, -2, 2$

23. $\dfrac{2x-1}{x}$

$x \neq -1/2, 0$

24. $\dfrac{x-3}{x+2}$

$x \neq -5, -2, 0$

The factor $(x + 5)$ cancels, so $x \neq -5$ appears nowhere in the answer.

25. $-\dfrac{1}{x(x+h)}$

$h \neq 0, x \neq 0, x \neq -h$

The main denominator is h, so the expression is undefined at $h = 0$. Simplifying removes h from the denominator, which is what lets you evaluate the limit as h approaches 0.

26. $\dfrac{x+1}{2x+1}$

$x \neq -1, -1/2, 0$

27. Let the one way distance be d. The total distance is $2d$ and the total time is $d/r_1 + d/r_2$, so the average speed is

$$\frac{2d}{\frac{d}{r_1}+\frac{d}{r_2}} = \frac{2}{\frac{1}{r_1}+\frac{1}{r_2}} = \frac{2r_1r_2}{r_1+r_2}$$

Algebraic exclusions: $r_1 \neq 0$, $r_2 \neq 0$, $r_1 + r_2 \neq 0$

A driving speed must be positive, so the situation itself requires $r_1 > 0$ and $r_2 > 0$, and those two conditions already imply the three algebraic exclusions.

The one way distance cancels. At 20 and 45 mph the average is 1800/65 ≈ 27.7 mph, not 32.5. The slower leg takes more time, so it carries more weight in the total. This expression is the harmonic mean of the two speeds.

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