Open-Top Box Design Challenge

Extension questions · answers at the back

Eight questions for students who finish early, or for a second day. Several stand on their own as short investigations. Answers begin on the last page.

1. On the 8.5 by 11 sheet, find **every** corner cut that gives a tray of exactly 63 in^3. Start by checking whether $x = 2$ works, then use that to factor the cubic. How many of your solutions can a real sheet produce?

2. A larger sheet measures 17 in by 22 in, which is exactly double the original in both directions. Predict the best corner cut and the volume **before** you compute anything, then check. State the rule you found as a general claim about scaling a sheet by a factor of k.

3. Take a square sheet of side s. Show that the corner cut giving the largest volume is always $x = s/6$, whatever s is, and that the volume there is $2s^3/27$.

4. You have 288 in^2 of card and you get to choose the shape of the rectangle. Compare 36 by 8, 24 by 12, 18 by 16, and the square. Which shape gives the biggest tray? Does the answer surprise you?

5. At the best cut on the original sheet, what percentage of the card ends up in the bin? Would a different cut waste less? Is wasting less the same as doing better?

6. Card stock costs 4 cents per square inch and you are charged for the whole sheet whether you use it or not. Which cut gives the most cubic inches per cent? Explain why this is the same optimisation problem you already solved.

7. Keep the short side at 8.5 in and let the long side grow: 11, 20, 50, 200, 1000. Work out the best cut for each. The answers approach a number. Say what it is and explain, using the factored form, why the long side stops mattering.

8. Design a brief of your own for the 8.5 by 11 sheet: two or three rules that make the best allowed cut come out at exactly 2 in. Swap with somebody and solve theirs.

Answers

1

$V(2) = 2 \cdot 7 \cdot 4.5 = 63$, so $x = 2$ is a solution and $(x - 2)$ is a factor. Solving $4x^3 - 39x^2 + 93.5x = 63$ and clearing the fraction gives $8x^3 - 78x^2 + 187x - 126 = 0$, which factors as $(x - 2)(8x^2 - 62x + 63) = 0$.

The quadratic gives $x = (31 \pm \sqrt{457})/8$, that is $x \approx 1.203$ and $x \approx 6.547$.

Three real solutions, and only two of them fit inside $0 < x < 4.25$. A cut of 6.547 in is longer than the sheet is wide. Two different trays hold exactly 63 in^3: one 8.59 by 6.09 with short walls, one 7 by 4.5 with 2 in walls.

2

Best cut 3.171 in, which is exactly double 1.5854. Volume 529.19 in^3, which is exactly eight times 66.148.

The claim: scaling a sheet by k scales the best cut by k and the volume by k^3. Every length in the model is multiplied by k, and volume is a product of three lengths.

3

$V(x) = x(s - 2x)^2 = 4x^3 - 4sx^2 + s^2x$. Setting the derivative $12x^2 - 8sx + s^2$ to zero gives $x = s/6$ or $x = s/2$. The second one closes the base up completely, so it is the minimum at the edge of the domain.

$V(s/6) = (s/6)(2s/3)^2 = 2s^3/27$. Without calculus: the same result follows from AM–GM applied to $4x$, $s - 2x$ and $s - 2x$, which are equal exactly when $x = s/6$.

4

36 by 8 gives 256.99 in^3, 24 by 12 gives 332.55, 18 by 16 gives 361.10, and the square (about 16.97 by 16.97) gives 362.04.

The square wins, and the ranking follows how close the rectangle is to square. Long thin sheets waste most of their length on a base that is too narrow to use.

5

At $x \approx 1.585$ the four squares total $4x^2 \approx 10.05$ in^2 out of 93.5 in^2, about 10.8 percent. At the 1.75 in design it is 12.25 in^2, about 13.1 percent.

Smaller cuts waste less card and hold less soil, so no. Minimising waste and maximising volume are different objectives, and the brief asked for one of them. A good answer names the trade-off instead of picking a side.

6

The sheet costs $93.5 \times 4 = 374$ cents no matter what you cut, so cubic inches per cent is $V(x)/374$. Dividing by a constant does not move the maximum, so the best cut is unchanged. If you were only charged for card that ends up in the tray, the objective would change and so would the answer.

7

Best cuts: 1.585, 1.852, 2.027, 2.102, 2.121 for long sides 11, 20, 50, 200, 1000. They approach 2.125, which is 8.5/4.

In $V = x(L - 2x)(8.5 - 2x)$ the first factor pair barely changes when L is huge, so maximising V is almost the same as maximising $x(8.5 - 2x)$, a quadratic with its vertex at $x = 8.5/4$. The short side decides the answer once the long side stops being scarce.

8

Answers vary. One that works: walls at least 2 in, and the base must fit an opening 4.5 in wide. That forces $x \geq 2$, the curve is falling there, so $x = 2$ wins with 63 in^3. Good briefs make the binding rule the one students would not guess.