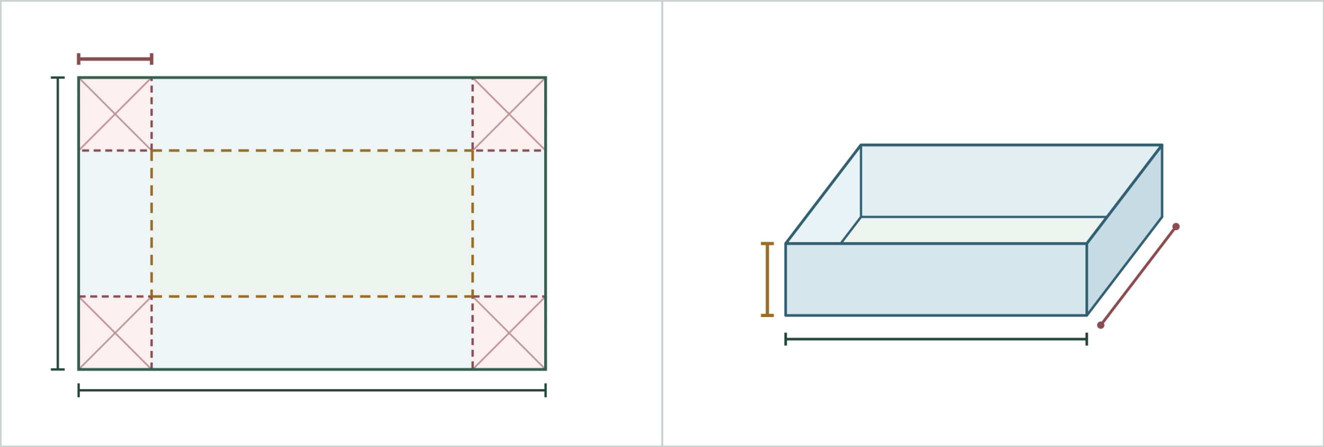
The Open-Top Box

Express edition · one class period · Name ____________________

What you are doing

Cut four equal squares out of the corners of a 16 in by 10 in sheet. Fold the sides up and tape the corners. You get a tray with no lid.

Find the corner cut that makes the tray hold the most and still fits on the shelf. The shelf opening is 5 in wide, so the short side of your tray cannot be more than 5 in.



Left: the flat sheet, corner squares removed, dashed lines are folds. Right: the same sheet folded up. Rose is thrown away, green is the base, blue folds up into the walls.

Step 1 · The three dimensions

Each corner square has side **x** inches. One is done for you.

	Dimension of the tray	In terms of x
a	How tall the walls are	**x**
b	The longer side of the base	16 − ______
c	The shorter side of the base	10 − ______

Hint

You cut a square off **each end** of every side, not just one end. Put your finger on the left edge of the flat drawing and slide it right, counting the squares you cross.

Step 2 · The volume formula

Volume of a box is length × width × height. Fill in the three brackets using your answers above.

V = (______________) × (______________) × (______________)

Step 3 · Try five cut sizes

The first row is done for you. Fill in the rest.

Cut x (in)	Longer base side	Shorter base side	Height	Volume (in^3)
1	14	8	1	14 × 8 × 1 = 112
1.5				
2				
2.5				
3				

Step 4 · The biggest tray

1. Circle the row with the **biggest volume**. Which cut is it, and how much does that tray hold?

Step 5 · Now check the shelf

The shelf opening is 5 in wide. Go back to your table and **cross out every row** where the shorter base side is bigger than 5 in, because those trays are too wide to go in.

2. Did the row you circled in Step 4 get crossed out?

3. Out of the rows that are **left**, which cut gives the biggest volume? This is your design.

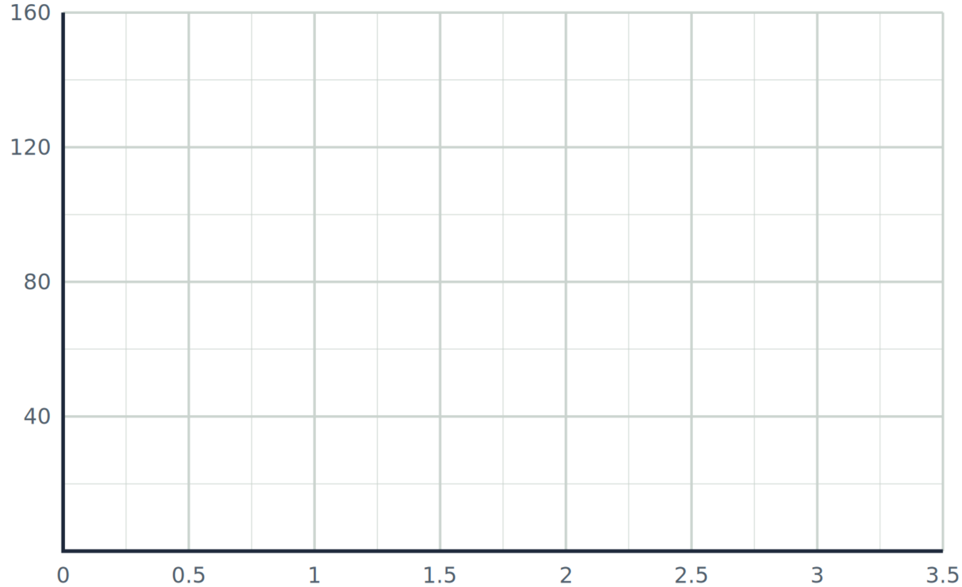
Step 6 · Explain your design

4. Somebody says “just cut 2 in squares, that holds the most.” They are right that it holds the most. Explain in one or two sentences why it is still the wrong answer here.

5. How many cubic inches did the shelf rule cost you? (Biggest volume − your volume.)

Draw the picture

Plot your five points from the table, then join them with a smooth curve.



Across: the cut size x, in inches. Up: the volume, in cubic inches.

6. After the highest point, does your curve go up or down?

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7. You only tested five cuts. Use your answer to question 6 to explain why no **other** allowed cut, such as 4 in or 3.5 in or anything else past your design, could beat it.

Bonus, if you finish early

1. Multiply your three brackets out and write V as $4x^3 -$ ______ $x^2 +$ ______ x. Check it by putting $x = 1$ in and seeing if you get 112.

2. What is the biggest square you could possibly cut out of this sheet before there is no tray left at all? Explain what the sheet looks like at that point.

3. The shelf gets rebuilt 6 in wide instead of 5 in. Does your design change? To what?

The Open-Top Box

Express edition · one class period · Teacher answer key

Burke Tutoring in Fremont
Polynomial modeling project

What this version is

A one-period version of the open-top box task, stripped to the single idea worth taking away: **the biggest answer is not always the allowed one.**

No inequality notation, no domain written formally, no estimation, no percentages. Students check the shelf rule by crossing rows out of a table rather than by solving $10 - 2x \le 5$. The longer packets remain available if you want the formal version later.

Runs in 25 to 35 minutes. Add 10 minutes if students build the tray.

Steps 1 and 2

Walls are **x** tall. Longer base side **16 − 2x**. Shorter base side **10 − 2x**. So $V = (16 - 2x)(10 - 2x)(x)$.

The near-universal error is writing $16 - x$ and $10 - x$. It is worth catching out loud: put both versions on the board, work out the base at $x = 2$ each way, and hold the folded tray up against a ruler.

Step 3 · The completed table

Cut x (in)	Longer base	Shorter base	Height	Volume (in³)	Fits 5 in shelf?
1	14	8	1	112	no, cross out
1.5	13	7	1.5	136.5	no, cross out
2	12	6	2	**144**	no, cross out
2.5	11	5	2.5	**137.5**	**yes**
3	10	4	3	120	yes

Every number is a whole number or a half. No calculator needed beyond the multiplication.

Questions 1 to 5

Q	Answer
1	$x = 2$ in, holding 144 in³.
2	Yes. Its shorter base side is 6 in, and the shelf is 5 in wide.
3	**x = 2.5 in, holding 137.5 in³.** Base 11 in by 5 in, walls 2.5 in. It fits the shelf with nothing to spare.
4	Looking for: a 2 in cut makes a tray 6 in wide, the shelf opening is only 5 in, so the tray will not go in. A tray that holds more but does not fit is worth nothing. Accept any wording that names both the number and the rule.
5	$144 - 137.5 = 6.5$ in³.

Questions 6 and 7 · the one that matters

6. Down. The curve rises to a single high point at $x = 2$ and falls after it.

7. Looking for: every allowed cut is 2.5 or bigger, so every allowed cut is past the top of the curve, on the part that is going down. Going further right only makes the tray smaller. So the best allowed cut is the smallest allowed cut, which is 2.5 in, and that covers cuts nobody tested.

This is the only genuinely abstract moment in the exercise. If a class is struggling, ask them to predict V at $x = 4$ from the shape of their curve before they compute it (it is 64), then ask whether they needed the arithmetic to know it would be small.

Bonus answers

	Answer
1	$V = 4x^3 - 52x^2 + 160x$. At $x = 1$: $4 - 52 + 160 = 112$. Matches the table.
2	Just under 5 in. At exactly 5 in the two cuts on the 10 in side meet in the middle, the base has no width left, and the sheet is four strips with nothing between them. Students often say 8 in, using the long side. Ask them to draw the tray at $x = 6$.
3	Yes. A 6 in opening lets the 2 in cut through, because that tray is 6 in wide. The design becomes $x = 2$ in with 144 in³, and the shelf stops mattering at all.

Running it in 30 minutes

0 to 5 min. Hold up two trays you cut earlier with visibly different corner sizes. Ask which holds more. Take a vote. Do not settle it.

5 to 15 min. Steps 1 to 3, in pairs. Circulate for the $16 - x$ error.

15 to 22 min. Steps 4 and 5. Let the disappointment land when the circled row gets crossed out. That reaction is the point of the exercise.

22 to 32 min. Questions 4 to 7, individually and in silence. Question 7 is the one to discuss as a class before the bell.

Where to go next

The poster board packet uses the same sheet and the same shelf, but adds inequality notation, the full nine-row table, the formal domain and a written defence. It is the natural follow-on a day or two later, and students who did this exercise already know the answer, which frees them to concentrate on the reasoning.

The brief

The garden club needs seedling trays. You are going to design one, and you get one piece of poster board to do it with.

Your job

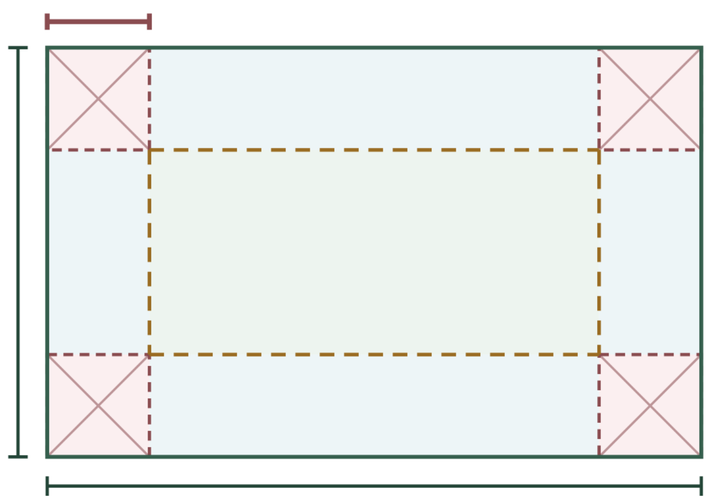
Cut four equal squares out of the corners of one 16 in by 10 in sheet, fold the flaps up, and tape the corners. That gives you an open-top tray.

Find the corner cut that makes the tray hold as much potting mix as possible **and still meets every rule below**.

Then convince somebody your answer is right. A number on its own is not an answer here.

The rules the tray has to follow

	Rule	Why it is there
R1	One sheet, 16 in by 10 in. All four corner squares the same size.	You get one sheet. No taping extra pieces on.
R2	Walls at least 1 in tall.	Shorter walls and the mix spills when someone slides the tray.
R3	The tray has to slide into the grow shelf. The opening is **5 in wide and 12 in deep**.	The shelf is already built. The tray has to fit it, not the other way round.
R4	Measure cuts to the nearest half inch.	That is as fine as a ruler and a pair of scissors get you on poster board.



Key

- Base of the tray
- The four sides that fold up
- Cut away and thrown out
- Fold along here
- Length of one corner cut, x

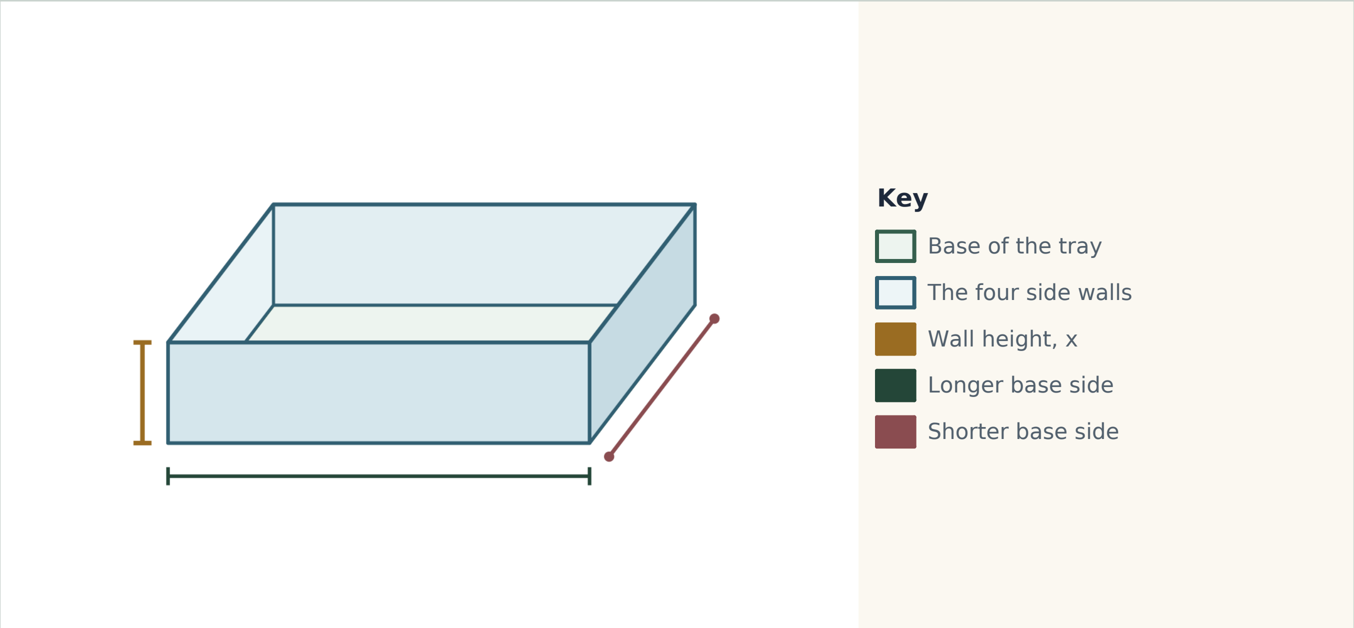
The flat sheet with the four corner squares taken out. The dashed lines are folds, not cuts.

Before you start

You need: poster board or a large sheet of card, a ruler, scissors, tape, a pencil. A calculator helps in Part C. You do not need graphing technology; every number in this packet can be worked out by hand.

Part A · Build the model

Call the side of each corner square **x** inches. Everything else about the tray depends on that one number.



The same sheet, folded up. Nothing has been added, only folded.

1. Fill in the three dimensions of the finished tray in terms of x.

Dimension	In terms of x
Height of the walls	
Longer side of the base	
Shorter side of the base	

Careful with this bit

A corner square is cut off **both** ends of every side. Trace one full edge of the drawing with your finger and count how many squares you cross before you write anything down.

2. Write the volume as a product of three factors. Do not expand it yet.

3. Now expand your answer to question 2 into standard form. Then check it: work out V(1) from the factored form and from the expanded form and make sure you get the same number both times.

4. Set your factored form equal to zero and list all three zeros. Two of them are cut sizes a real sheet could produce. One is not. Say which is which, and why.

Part B · Say what x is allowed to be

A cubic will happily take $x = 20$ or $x = -3$. A sheet of poster board will not. This part is where you draw the line between the two.

5. What goes wrong if $x = 0$? Describe the object you would be holding.

6. The shorter side of the sheet is 10 in. How large can x get before the tray stops existing? Show the inequality you solved. Check the 16 in side too, and say which of the two runs out first.

7. Write the domain of V that comes from the sheet alone. Rules R2, R3 and R4 do not count yet. Use inequality notation.

Two different questions

The domain of the *function* $4x^3 - 52x^2 + 160x$ is every real number. The domain of the *model* is much smaller. You are answering the second one.

Now bring in the design rules

8. Rule R2 says the walls are at least 1 in tall. Turn that into an inequality in x.

9. Rule R3 says the tray has to fit an opening 5 in wide and 12 in deep. The base measures $(16 - 2x)$ by $(10 - 2x)$. Write one inequality for the width fit and one for the depth fit, then solve both.

10. Put questions 7, 8 and 9 together. Write the one interval of x values that satisfies everything at once. This is your **feasible interval**.

Part C · Test every allowed cut size

Rule R4 says cuts go to the nearest half inch, so there are only nine cut sizes worth writing down. Fill the whole table in, including the ones that break a rule, because you will need them in Part D.

x (in)	Height	Longer base side	Shorter base side	Volume (in^3)	Passes R2 and R3?
0.5					
1.0					
1.5					
2.0					
2.5					
3.0					
3.5					
4.0					
4.5					

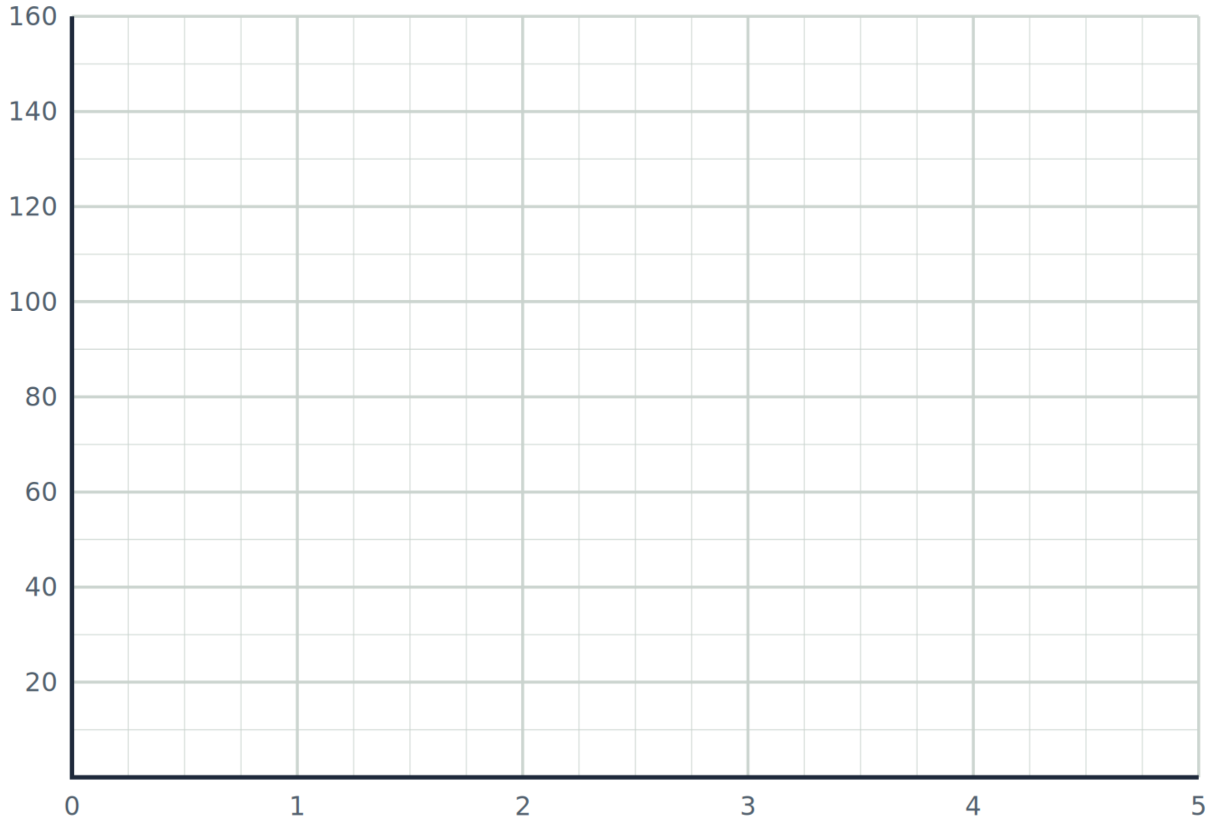
11. Which cut size gives the biggest volume of all nine?

12. Which cut size gives the biggest volume **out of the ones that pass every rule**? Are your answers to 11 and 12 the same?

13. Two different cut sizes in your table give volumes that are close to each other, but the two trays look nothing alike. Find a pair and describe how the trays differ.

Part D · Graph it

Plot all nine points from your Part C table, then join them with a smooth curve. Stop the curve at the ends of the domain you found in question 7. Do not run it past them.



Horizontal axis: corner cut x, in inches. Vertical axis: volume, in cubic inches.

14. Shade the part of the horizontal axis that the shelf rule throws out. Then mark your feasible interval from question 10 on the same axis.

15. Mark the highest point on the whole curve and label it P. Mark the highest point that is inside the shaded-in feasible interval and label it Q. Are P and Q the same point?

16. Look at your curve to the right of P. Is it going up, going down, or doing both? Answer in one sentence.

Part E · Check your design against every rule

Write your chosen cut size at the top, then run it through all four rules. If it fails one, go back and choose again.

My chosen corner cut is x = _______________ in

Rule	The number I have to check	My value	Pass?
R1	All four corner squares equal, one sheet		
R2	Wall height, needs to be at least 1 in		
R3	Base width, needs to be at most 5 in		
R3	Base depth, needs to be at most 12 in		
R4	Cut is a whole number of half inches		

17. What volume does your tray hold?

18. How much volume did the shelf rule cost you, compared with the best cut on the whole curve? Give the loss in cubic inches and as a percentage.

Part F · Defend the design

This is the part that gets read most carefully. Write in full sentences, and point at your own numbers when you make a claim.

19. State your design in one sentence: the cut, the finished dimensions, and the volume.

20. Somebody proposes a cut that gives a bigger volume than yours. Give the number they almost certainly picked, and explain exactly which rule it breaks and by how much.

21. Explain why no allowed cut beats yours. Your answer to question 16 is the key: if the curve only goes down after the peak, and every allowed cut is past the peak, what does that tell you about where the best allowed cut has to be?

22. Suppose the grow shelf were rebuilt 6 in wide instead of 5 in. Would your design change? Say what the new answer would be and why.

Build it and see

The model says one thing. The poster board gets a vote too.

23. Cut and fold your tray. Measure the finished inside dimensions with a ruler and multiply them out.

	Predicted	Measured
Wall height		
Longer base side		
Shorter base side		
Volume		

24. Your measured volume is probably smaller than your predicted volume. Give two physical reasons why.

25. Fill the tray with rice or dry beans, pour it into a measuring cup, and record the volume in cups. One cup is about 14.4 in^3. How close did you get?

Open-Top Box Design Challenge

Poster board edition · 16 in by 10 in sheet · Teacher answer key

Burke Tutoring in Fremont
Polynomial modeling project

What changed from the letter-sheet version

Same construction, same design brief, easier arithmetic. On a 16 in by 10 in sheet the maximum lands on a whole number, so the two rounds of estimation disappear. Rule R4 moves from eighths to half inches, which leaves exactly nine cut sizes to check the whole of Part C fits on one page.

What did **not** change: the shelf rule still pushes the peak out of the feasible interval, so students still have to notice that the best answer is not allowed, and still have to argue over an interval rather than off a table. Here the constraint costs 4.5 percent of the volume rather than 0.8 percent, which students find much easier to care about.

The model

$V(x) = x(16 - 2x)(10 - 2x) = 4x^3 - 52x^2 + 160x$

Zeros at $x = 0$, $x = 5$ and $x = 8$. Only the first two are reachable on a real sheet.

Domain from the sheet alone: $0 < x < 5$.

Peak of the curve: **$x = 2$ in, $V = 144$ in³**, and it is exact. $V'(x) = 12x^2 - 104x + 160 = 4(x - 2)(3x - 20)$, so the critical values are $x = 2$ and $x = 20/3 \approx 6.67$, and only $x = 2$ is in the domain.

Feasible interval once the brief is applied: **$2.5 \le x < 5$**.

The design: **$x = 2.5$ in, base 11 in by 5 in, walls 2.5 in, $V = 137.5$ in³**.

Part C, completed

x (in)	Height	Base (in)	Volume (in³)	R2?	R3 width	R3 depth	Passes?
0.5	0.5	15 × 9	67.5	no	no	no	no
1.0	1.0	14 × 8	112	yes	no	no	no
1.5	1.5	13 × 7	136.5	yes	no	no	no
2.0	2.0	12 × 6	**144**	yes	no	yes	no
2.5	2.5	11 × 5	**137.5**	yes	yes	yes	**yes**
3.0	3.0	10 × 4	120	yes	yes	yes	yes
3.5	3.5	9 × 3	94.5	yes	yes	yes	yes
4.0	4.0	8 × 2	64	yes	yes	yes	yes
4.5	4.5	7 × 1	31.5	yes	yes	yes	yes

Every volume in the table is a whole number or a half. No student needs a calculator for more than the multiplication.

Answers, question by question

Part A

Q	Answer
1	Height x, longer base side $16 - 2x$, shorter base side $10 - 2x$.
2	$V(x) = x(16 - 2x)(10 - 2x)$.
3	$V(x) = 4x^3 - 52x^2 + 160x$. Check at $x = 1$: factored gives $1 \cdot 14 \cdot 8 = 112$, expanded gives $4 - 52 + 160 = 112$.
4	$x = 0$, 5 and 8. At $x = 0$ nothing is cut and there is no tray. At $x = 5$ the 10 in side has closed to zero width. $x = 8$ would mean cutting 16 in off a 10 in side, which no sheet can do.

Part B

Q	Answer
5	A flat sheet. No walls, no volume.
6	$10 - 2x > 0$ gives $x < 5$. The 16 in side gives $x < 8$. The short side runs out first, so 5 is the one that binds.
7	$0 < x < 5$.
8	$x \geq 1$.
9	Width fit: $10 - 2x \leq 5$ gives $\mathbf{x \geq 2.5}$. Depth fit: $16 - 2x \leq 12$ gives $x \geq 2$. Both have to hold.
10	$2.5 \leq x < 5$. The shelf width is the binding rule; the wall rule and the depth fit are already satisfied by everything in that interval.

Parts C and D

Q	Answer
11	$x = 2$ in, with 144 in^3.
12	$x = 2.5$ in, with 137.5 in^3. No, they are not the same, and noticing that is the whole project.
13	The clean pair is $x = 1.5$ (136.5 in^3) and $x = 2.5$ (137.5 in^3), a difference of one cubic inch. The first tray is 13 by 7 with 1.5 in walls, wide and shallow; the second is 11 by 5 with 2.5 in walls, narrower and deeper. Almost the same amount of soil, very different tray. Some students offer 1.0 and 3.0 instead (112 and 120), which makes the same point less sharply.
15	P is the peak at (2, 144). Q is at (2.5, 137.5), at the left end of the feasible interval. They are different points.
16	It only goes down. Students who say “both” have usually run the curve past $x = 5$ and should be sent back to question 7.

Parts E and F

Q	Answer
Design	$x = 2.5$ in. Base 11 in by 5 in, walls 2.5 in, volume 137.5 in³. The base width is 5 in exactly, so the tray fits the opening with nothing to spare.
17	137.5 in³.
18	$144 - 137.5 = 6.5$ in³, and $6.5 \div 144 \approx 4.5$ percent.
20	They picked $x = 2$ in, which holds 144 in³. That tray has a base 12 in by 6 in. The shelf opening is 5 in wide, so it is a full inch too wide and fails R3. Worth noticing: its 12 in depth fits the shelf exactly, so a student who only checks the depth will think it passes.
21	The argument, in the form students should reach: the curve rises to a single peak at $x = 2$ and falls after it. Every allowed cut is 2.5 or more, so every allowed cut is past the peak, on the falling part. That means the largest allowed volume has to be at the smallest allowed cut, which is 2.5. This covers cut sizes nobody tested, because it is a statement about the shape of the curve rather than a list of values.
22	Yes, it changes. A 6 in opening gives $10 - 2x \le 6$, so $x \ge 2$, and the depth rule also gives $x \ge 2$. The feasible interval becomes $2 \le x < 5$, and the peak at $x = 2$ is now inside it. So the answer becomes $x = 2$ in with 144 in³, and the constraint stops deciding the design at all.

Part G

Measured volumes usually land 4 to 8 percent under the prediction. The two reasons students should reach: poster board has thickness, so each fold happens over a strip rather than on a line and eats into the base; and taped walls lean inward slightly at the top. A tray that measures 10.9 by 4.9 by 2.45 holds about 130.9 in³, which is a typical result.

Where students get stuck

Subtracting one x instead of two. $V = x(16 - x)(10 - x)$ is the most common wrong model. Send them back to the drawing to trace an edge.

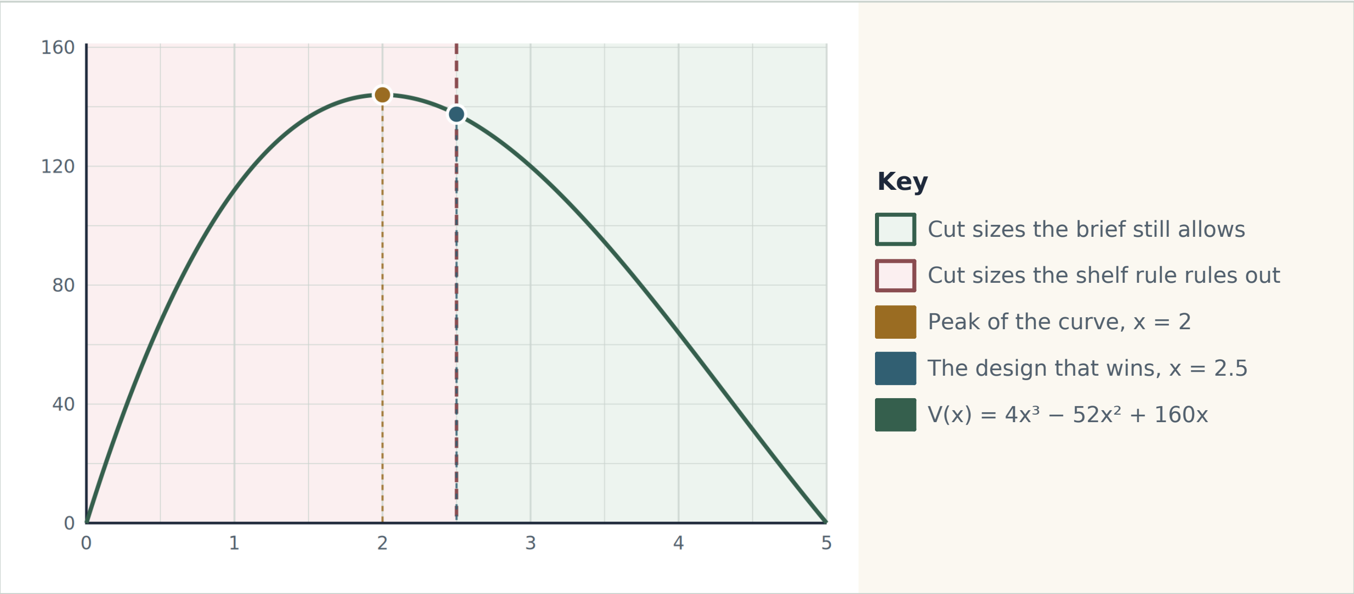
Using the long side for the domain. Some write $0 < x < 8$. Ask what the tray looks like at $x = 6$ and let them try to draw it.

Stopping at the peak. This is the failure the brief is built to catch. Do not warn them in advance. Let Part E catch it and let them fix it.

Checking the depth and forgetting the width. At $x = 2$ the depth fits exactly, so this error looks like a pass. Ask for both inequalities every time.

Arguing from the table alone. “I checked all nine” is true here, which makes question 21 harder to motivate than it looks. Push them to answer it without the table: what if cuts could be any size at all, not just halves?

The graph, for projecting



The gap between the gold point and the blue point is what the shelf rule costs. On this sheet it is 6.5 in³, which is visible from the back of the room.

Running it

When	What
First 10 min	Hand each pair a sheet and let them cut a tray with any corner size they like. No mathematics yet. Stand three visibly different trays at the front.
Next 25 min	Parts A and B. Stop the class when the first pair writes $16 - x$ and put both versions on the board without saying which is right.
Next 20 min	Parts C and D. The table is short enough to finish in class, which is the main reason this version fits in less time than the letter-sheet one.
Last 20 min	Parts E and F. The moment a pair works out that a 2 in cut does not fit the shelf is the moment the project starts working. Part F in silence, individually.
Homework	Part G, the build and measure, if you did not get to it.

If a pair finishes early

Ask them for **every** cut that gives exactly 112 in³. They already know $x = 1$ works from their table, so $(x - 1)$ is a factor; dividing leaves $x^2 - 12x + 28$, giving $x = 6 \pm 2\sqrt{2}$, that is 3.172 and 8.828. Three roots, two of which a real sheet can produce. Same question with 120 in³ gives $x = 3$ and $x = 5 \pm \sqrt{15}$, that is 1.127 and 8.873.

The rubric works with this version unchanged, since its criteria are written to fit any of the three sheet sizes. The extension questions on this site now use the same 16 in by 10 in sheet as this packet, so the numbers line up directly.

Rubric

Five criteria, four points each, twenty points total. The written defence is weighted the same as the algebra on purpose.

	4 · Exemplary	3 · Proficient	2 · Developing	1 · Beginning
The model	V(x) written as a correct product and expanded correctly, with the three dimensions justified from the drawing.	V(x) correct in both forms. Justification is thin or missing.	One structural error, usually subtracting a single x, carried through consistently.	The function does not describe the tray in the picture.
Domain and constraints	States the correct domain with a reason, converts every rule in the brief into an inequality, and identifies the correct feasible interval.	Domain correct and most rules converted. One inequality missing or reversed.	Domain stated without reasoning, or only one rule handled.	No distinction between what the function allows and what a sheet allows.
Evidence: table and graph	Table complete and accurate, graph plotted on the domain with the peak and the feasible interval both marked. If the version being used calls for refinement, it is narrowed to at least two decimal places.	Table and graph complete with minor arithmetic slips. Refinement present where the version requires it.	Table has gaps, or the graph runs outside the domain, or the required refinement is missing.	Little numerical or graphical evidence.
The design decision	Chooses the correct cut size, checks it against every rule in the brief, and reports the volume and the cost of the binding constraint in both cubic units and percent.	Chooses the correct cut and checks most rules. Cost of the constraint missing or miscomputed.	Chooses the peak of the curve and does not notice it fails a rule.	No clear design, or a cut size that breaks a rule with no comment.
Argument and precision	Explains why no allowed cut beats the chosen one, using the shape of the curve over the whole feasible interval rather than only the tested values. Names the tempting wrong answer and the rule it breaks. Units and the required precision are used throughout.	Clear argument resting mainly on the table rather than the whole interval. Units mostly present.	Restates the answer without an argument, or argues only from tested points.	No defence, or a defence that contradicts the student's own numbers.

Criterion	Score
The model	
Domain and constraints	
Evidence: table and graph	
The design decision	
Argument and precision	
Total out of 20	

Before you hand it in

Tick each line only if you can point at the place in your packet where it happens. If you cannot find it, it is not done.

- ○ My volume function is a product of three factors and I can say where each factor comes from on the drawing.
- ○ I expanded it and my expanded form gives the same value as my factored form for at least one x I tested.
- ○ I wrote the domain as an inequality and I said what goes wrong at each end.
- ○ I turned every rule in the brief into an inequality, not just the one about the walls.
- ○ My graph stops at the ends of the domain instead of running off the grid.
- ○ I marked the peak of the curve and the best allowed cut as two separate points.
- ○ My chosen cut is measured to the precision my worksheet asks for.
- ○ I checked the base width and the base depth against the shelf, separately.
- ○ I said how much volume the shelf rule cost me, in cubic inches and as a percentage.
- ○ My defence explains why every allowed cut loses to mine, not just the ones in my table.
- ○ I named the cut somebody would wrongly pick and the rule it breaks.
- ○ Every number in my packet has a unit attached to it.

The one that decides the grade

Find the question in your worksheet that asks you to defend your design, and read your answer to it again. If that answer would still be true with the shelf rule deleted, you have not answered it.

Open-Top Box Design Challenge

Extension questions · answers at the back

Eight questions for students who finish early, or for a second day. Several stand on their own as short investigations. Answers begin on the last page.

1. On the 8.5 by 11 sheet, find **every** corner cut that gives a tray of exactly 63 in^3. Start by checking whether $x = 2$ works, then use that to factor the cubic. How many of your solutions can a real sheet produce?

2. A larger sheet measures 17 in by 22 in, which is exactly double the original in both directions. Predict the best corner cut and the volume **before** you compute anything, then check. State the rule you found as a general claim about scaling a sheet by a factor of k.

3. Take a square sheet of side s. Show that the corner cut giving the largest volume is always $x = s/6$, whatever s is, and that the volume there is $2s^3/27$.

4. You have 288 in^2 of card and you get to choose the shape of the rectangle. Compare 36 by 8, 24 by 12, 18 by 16, and the square. Which shape gives the biggest tray? Does the answer surprise you?

5. At the best cut on the original sheet, what percentage of the card ends up in the bin? Would a different cut waste less? Is wasting less the same as doing better?

6. Card stock costs 4 cents per square inch and you are charged for the whole sheet whether you use it or not. Which cut gives the most cubic inches per cent? Explain why this is the same optimisation problem you already solved.

7. Keep the short side at 8.5 in and let the long side grow: 11, 20, 50, 200, 1000. Work out the best cut for each. The answers approach a number. Say what it is and explain, using the factored form, why the long side stops mattering.

8. Design a brief of your own for the 8.5 by 11 sheet: two or three rules that make the best allowed cut come out at exactly 2 in. Swap with somebody and solve theirs.

Answers

1

$V(2) = 2 \cdot 7 \cdot 4.5 = 63$, so $x = 2$ is a solution and $(x - 2)$ is a factor. Solving $4x^3 - 39x^2 + 93.5x = 63$ and clearing the fraction gives $8x^3 - 78x^2 + 187x - 126 = 0$, which factors as $(x - 2)(8x^2 - 62x + 63) = 0$.

The quadratic gives $x = (31 \pm \sqrt{457})/8$, that is $x \approx 1.203$ and $x \approx 6.547$.

Three real solutions, and only two of them fit inside $0 < x < 4.25$. A cut of 6.547 in is longer than the sheet is wide. Two different trays hold exactly 63 in³: one 8.59 by 6.09 with short walls, one 7 by 4.5 with 2 in walls.

2

Best cut 3.171 in, which is exactly double 1.5854. Volume 529.19 in³, which is exactly eight times 66.148.

The claim: scaling a sheet by k scales the best cut by k and the volume by k^3. Every length in the model is multiplied by k, and volume is a product of three lengths.

3

$V(x) = x(s - 2x)^2 = 4x^3 - 4sx^2 + s^2x$. Setting the derivative $12x^2 - 8sx + s^2$ to zero gives $x = s/6$ or $x = s/2$. The second one closes the base up completely, so it is the minimum at the edge of the domain.

$V(s/6) = (s/6)(2s/3)^2 = 2s^3/27$. Without calculus: the same result follows from AM–GM applied to $4x$, $s - 2x$ and $s - 2x$, which are equal exactly when $x = s/6$.

4

36 by 8 gives 256.99 in³, 24 by 12 gives 332.55, 18 by 16 gives 361.10, and the square (about 16.97 by 16.97) gives 362.04.

The square wins, and the ranking follows how close the rectangle is to square. Long thin sheets waste most of their length on a base that is too narrow to use.

5

At $x \approx 1.585$ the four squares total $4x^2 \approx 10.05$ in² out of 93.5 in², about 10.8 percent. At the 1.75 in design it is 12.25 in², about 13.1 percent.

Smaller cuts waste less card and hold less soil, so no. Minimising waste and maximising volume are different objectives, and the brief asked for one of them. A good answer names the trade-off instead of picking a side.

6

The sheet costs $93.5 \times 4 = 374$ cents no matter what you cut, so cubic inches per cent is $V(x)/374$. Dividing by a constant does not move the maximum, so the best cut is unchanged. If you were only charged for card that ends up in the tray, the objective would change and so would the answer.

7

Best cuts: 1.585, 1.852, 2.027, 2.102, 2.121 for long sides 11, 20, 50, 200, 1000. They approach 2.125, which is 8.5/4.

In $V = x(L - 2x)(8.5 - 2x)$ the first factor pair barely changes when L is huge, so maximising V is almost the same as maximising $x(8.5 - 2x)$, a quadratic with its vertex at $x = 8.5/4$. The short side decides the answer once the long side stops being scarce.

8

Answers vary. One that works: walls at least 2 in, and the base must fit an opening 4.5 in wide. That forces $x \geq 2$, the curve is falling there, so $x = 2$ wins with 63 in^3. Good briefs make the binding rule the one students would not guess.

When to use this

The same construction on a much larger sheet. The maximum is irrational, so there is no tidy answer to land on, and question 4 asks students to divide a cubic by a linear factor they have to find first.

I use this with students who finished the shorter versions quickly and were not stretched by them. It suits an honours group, or a precalculus class revisiting polynomials before optimisation.

The problem

A flat sheet of corrugated card measures 36 in by 24 in. Cut four equal squares from the corners, fold the sides up, and tape them. The construction is the same as the poster board version; the numbers are not.

1. Write V(x) in factored and standard form, and state the domain.

2. Find the exact value of x that maximises the volume. Leave your answer as a radical, then give it to three decimal places. Both critical values come out of the same quadratic, so explain how you know which one to discard.

3. Find the exact maximum volume, in the form $a + b\sqrt{c}$.

4. Find **every** corner cut that gives a volume of exactly 1620 in^3. Test $x = 3$ first, divide the cubic by the factor that gives you, then solve what is left. How many of your answers are physically possible?

5. The finished tray has to fit a pallet slot 15 in wide. Write the inequality, find the feasible interval, and say whether the unconstrained maximum survives it. Compare what happens here with what happened on the 16 in by 10 in sheet.

6. A supplier offers you 864 in² of card in any rectangle you like, and 36 by 24 is one option. Investigate at least three others and say which shape you would order.

Answers

1. $V(x) = x(36 - 2x)(24 - 2x) = 4x^3 - 120x^2 + 864x$, on the domain $0 < x < 12$.

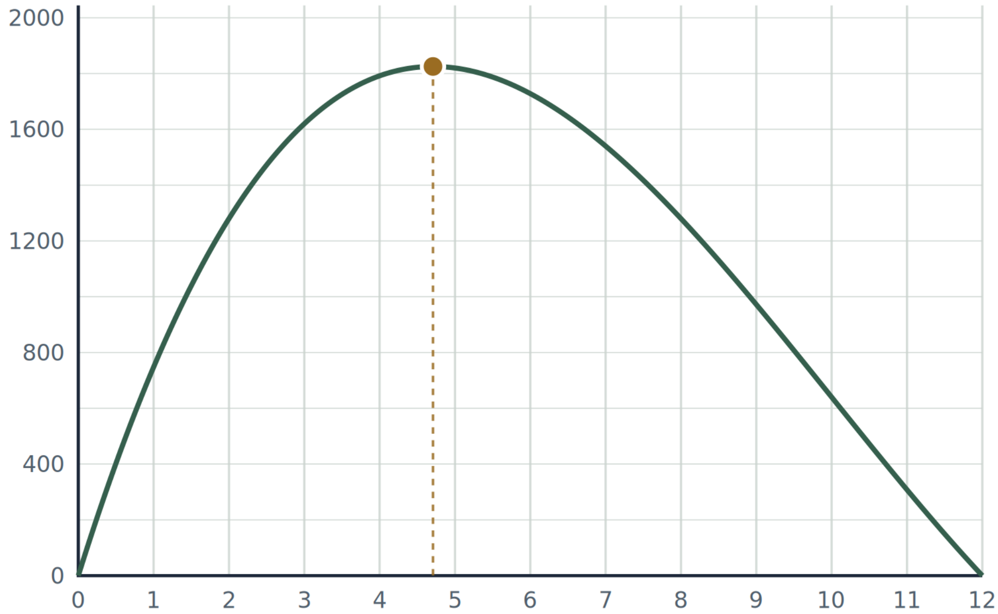
2. $V'(x) = 12x^2 - 240x + 864 = 12(x^2 - 20x + 72)$, so $x = 10 \pm 2\sqrt{7}$. The root $10 + 2\sqrt{7} \approx 15.29$ lies outside $0 < x < 12$, so the maximum is at $x = 10 - 2\sqrt{7} \approx 4.708$ in. Students who test a value either side, and students who notice the cubic must return to zero at $x = 12$, both reach this.

3. $V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825.297$ in³.

4. $V(3) = 3 \cdot 30 \cdot 18 = 1620$, so $(x - 3)$ is a factor. Dividing $4x^3 - 120x^2 + 864x - 1620$ by 4 and then by $(x - 3)$ leaves $x^2 - 27x + 135$, giving $x = (27 \pm 3\sqrt{21})/2$, that is 6.626 and 20.374. Three real solutions, two of them physical: at $x = 20.374$, twice that is more than the 24 in side, so the two cuts on that side would overlap and no real sheet can produce it.

5. $24 - 2x \le 15$ gives $x \ge 4.5$, so the feasible interval is $4.5 \le x < 12$. The maximum at 4.708 survives, because it is already above 4.5. This is the opposite of what happens on the 16 by 10 sheet, where the same kind of rule pushes the maximum out of range, and asking students to account for the difference is worth more than either result on its own.

6. Rectangles of area 864 in²: 48 by 18 gives 1600 in³, 36 by 24 gives 1825, 32 by 27 gives 1871, and the square of side about 29.39 in gives 1881. The closer to square, the larger the tray, which matches the 160 in² investigation on the extension sheet.



The volume curve for the 36 in by 24 in sheet on 0 < x < 12, with the maximum marked in gold.

The brief

The garden club needs seedling trays. You are going to design one, and you only get one sheet of card stock to do it with.

Your job

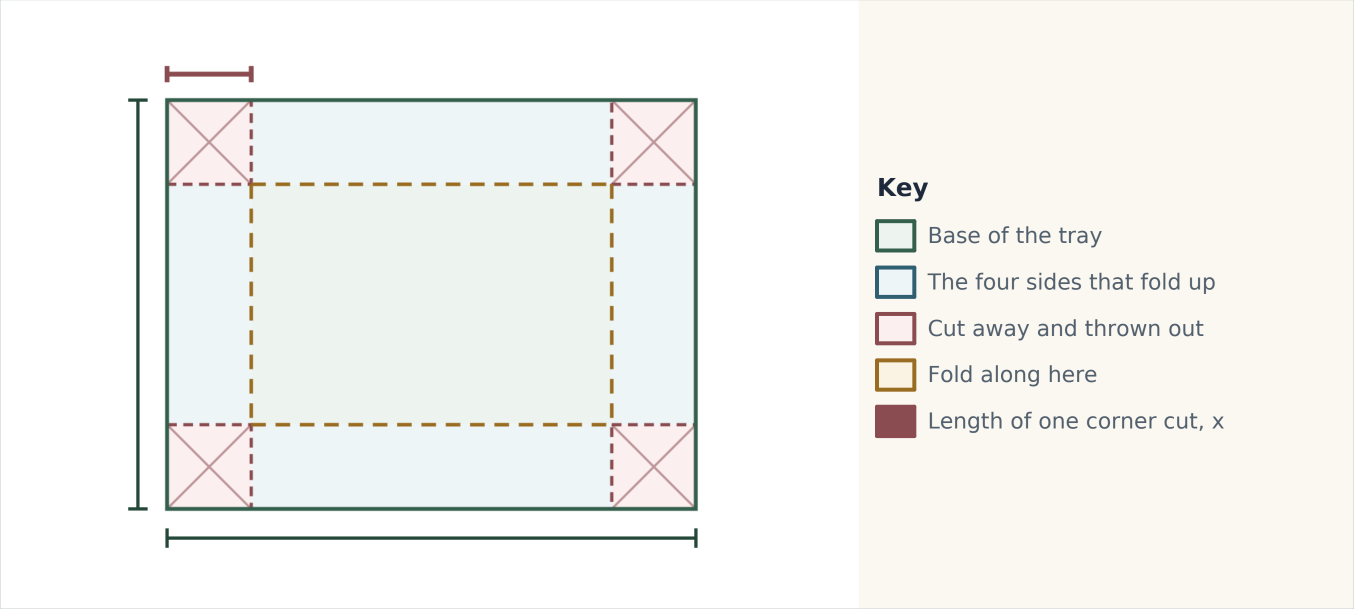
Cut four equal squares out of the corners of one 8.5 in by 11 in sheet, fold the flaps up, and tape the corners. That gives you an open-top tray.

Find the corner cut that makes the tray hold as much potting mix as possible **and still meets every rule below**.

Then convince somebody your answer is right. A number on its own is not an answer here.

The rules the tray has to follow

	Rule	Why it is there
R1	One sheet, 8.5 in by 11 in. All four corner squares the same size.	You get one sheet. No taping extra pieces on.
R2	Walls at least 1 in tall.	Shorter walls and the mix spills when someone slides the tray.
R3	The tray has to slide into the grow shelf. The opening is **5 in wide and 8 in deep**.	The shelf is already built. The tray has to fit it, not the other way round.
R4	Measure cuts to the nearest 1/8 in.	That is as fine as a ruler and a pair of scissors get you.



The flat sheet with the four corner squares taken out. The dashed lines are folds, not cuts.

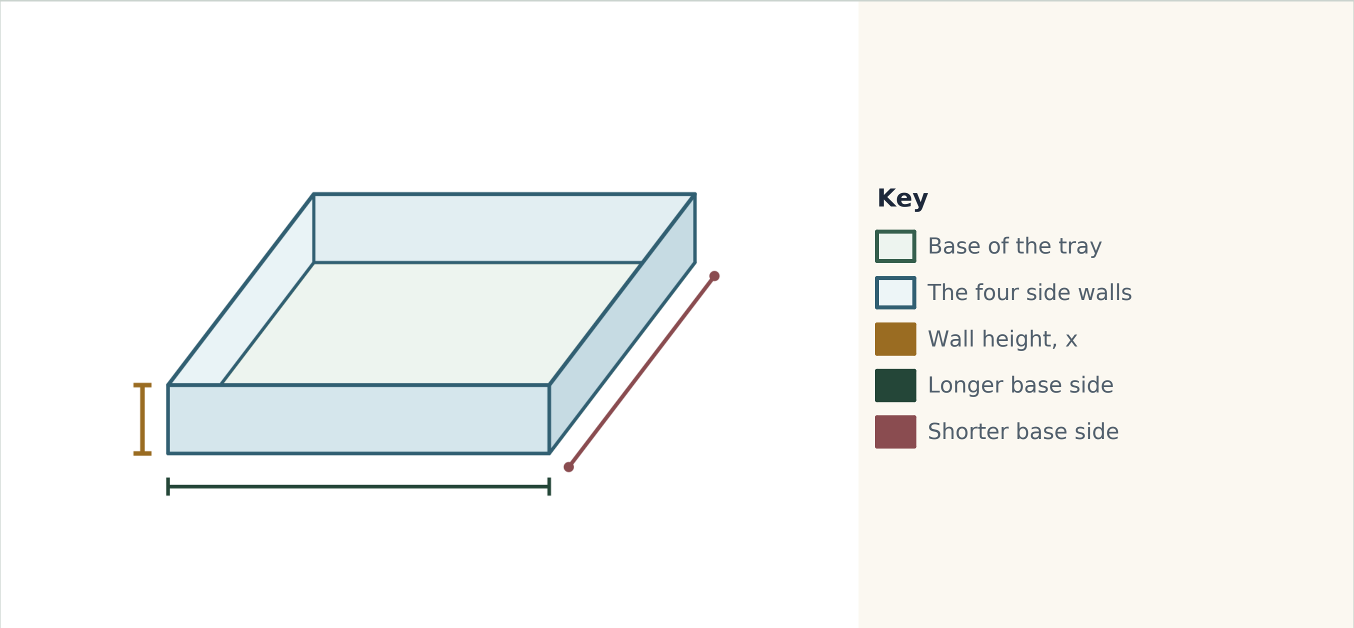
What you hand in

Parts A through G of this packet, filled in, plus the tray you actually built. Part G is the one that carries the most weight, so leave yourself time for it.

> **Before you start**
>
> You need: card stock or a manila folder, a ruler with eighths, scissors, tape, a pencil. A calculator helps in Part C. Graphing technology is optional everything here can be done by hand.

Part A · Build the model

Call the side of each corner square **x** inches. Everything else on the tray depends on that one number.



The same sheet, folded. Nothing has been added only folded.

1. Fill in the three dimensions of the finished tray in terms of x.

Dimension	In terms of x
Height of the walls	
Longer side of the base	
Shorter side of the base	

2. Write the volume as a product of three factors. Do not expand it yet.

3. Now expand your answer to question 2 into standard form.

4. What is the degree of V(x)? What is the leading coefficient? Without graphing anything, what does the leading coefficient tell you about the two ends of the curve?

5. Set your factored form equal to zero and list all three zeros. Two of them are cut sizes you could physically make. One is not. Say which is which and why.

Part B · Say what x is allowed to be

A cubic will happily take $x = 20$ or $x = -3$. A sheet of card will not. This part is where you draw the line between the two.

6. What goes wrong if $x = 0$? Describe the object you would be holding.

7. The shorter side of the sheet is 8.5 in. What is the largest x can get before the tray stops existing? Show the inequality you solved.

8. Write the domain of V that comes from the sheet alone rules R2, R3 and R4 do not count yet. Use inequality notation.

Careful here

The domain of the *function* $4x^3 - 39x^2 + 93.5x$ is every real number. The domain of the *model* is much smaller. Those are two different questions and you are answering the second one.

Now bring in the design rules

9. Rule R2 says the walls are at least 1 in tall. Turn that into an inequality in x.

10. Rule R3 says the tray has to fit an opening 5 in wide and 8 in deep. The base measures $(11 - 2x)$ by $(8.5 - 2x)$. Write one inequality for the width fit and one for the depth fit, then solve both.

11. Put questions 8, 9 and 10 together. Write the one interval of x values that satisfies everything at once. This is your **feasible interval**.

Part C · Test some cut sizes

Before you graph anything, get a feel for the numbers. Fill the table in. Round volumes to two decimal places.

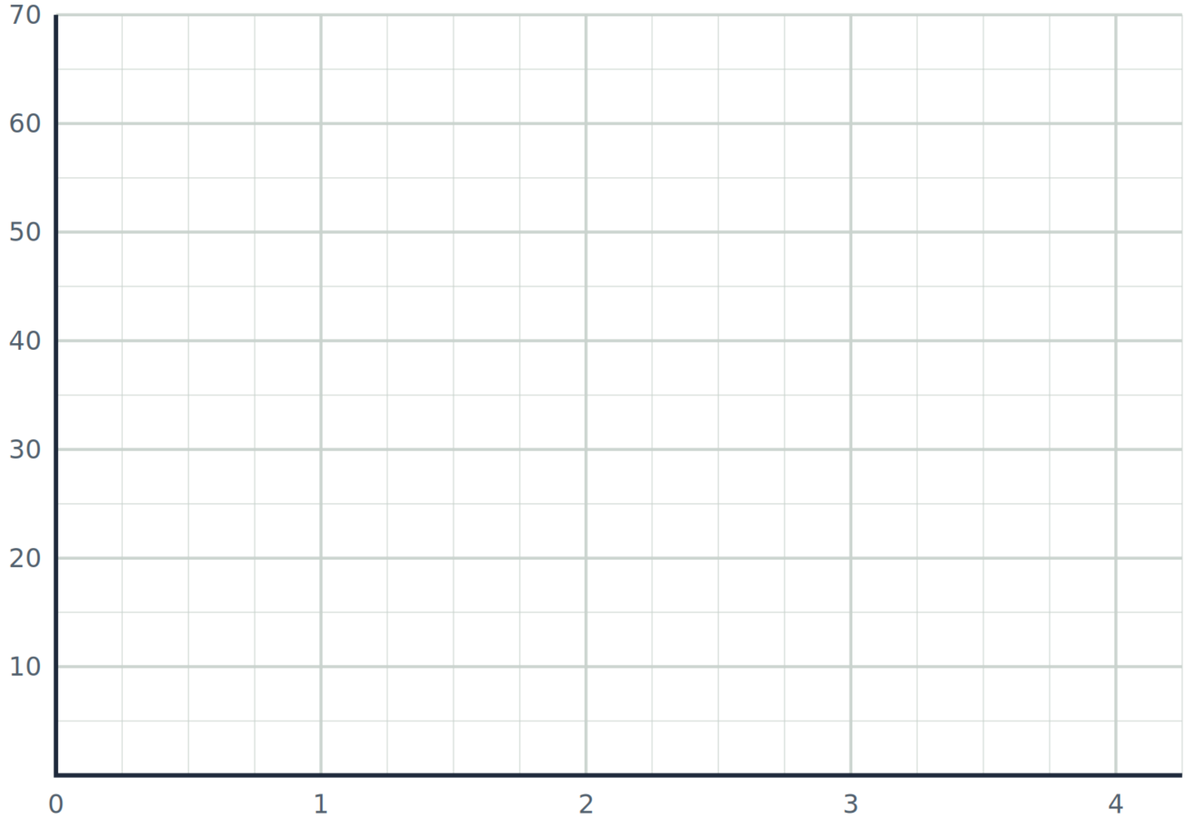
x (in)	Height	Longer base side	Shorter base side	Volume (in³)	Passes R2 and R3?
0.25					
0.50					
0.75					
1.00					
1.25					
1.50					
1.75					
2.00					
2.25					
2.50					
3.00					
3.50					
4.00					

12. Between which two of your x values does the volume stop climbing and start dropping?

13. Two different cut sizes in your table give volumes within a cubic inch of each other, but the trays look nothing alike. Find a pair and describe how the two trays differ.

Part D · Graph it

Plot every point from your Part C table, then join them with a smooth curve. Stop the curve at the ends of the domain you found in question 8. Do not run it past them.



Horizontal axis: corner cut x, in inches. Vertical axis: volume, in cubic inches.

14. Shade the part of the horizontal axis that rules R2 and R3 throw out. Then mark the feasible interval from question 11 on the same axis.

15. Mark the highest point on the curve and label it P. Mark the highest point that is inside the feasible interval and label it Q. Are P and Q the same point?

Part E · Pin down the maximum

Your graph gives you a rough answer. Now sharpen it with a table. Pick a starting range from your graph, then narrow it twice.

Round 1 step of 0.10

x						
V(x)						

Round 2 step of 0.02

x						
V(x)						

16. From your two rounds, the volume peaks near x = ______ in, giving V ≈ ______ in³. State how confident you are in each decimal place and why.

17. Rule R4 says you can only cut to the nearest 1/8 in. List the eighths on either side of your answer to question 16 and work out the volume at each one.

18. Does the peak of the curve sit inside your feasible interval, or outside it? Answer in one sentence, then say what that means for your design.

Part F · Check the design against every rule

Write your chosen cut size at the top, then run it through all four rules. If it fails one, go back and choose again.

My chosen corner cut is x = ______________ in

Rule	The number I have to check	My value	Pass?
R1	All four corner squares equal, one sheet		
R2	Wall height, needs to be at least 1 in		
R3	Base width, needs to be at most 5 in		
R3	Base depth, needs to be at most 8 in		
R4	Cut written as a whole number of eighths		

19. What volume does your tray hold?

20. How much volume did the shelf rule cost you, compared with the very best cut on the curve? Give the loss in cubic inches and as a percentage.

Part G · Defend the design

This is the part that gets read most carefully. Write in full sentences. Point at your own numbers when you make a claim.

21. State your design in one sentence: the cut, the finished dimensions, and the volume.

22. Explain why no other cut size that follows all four rules beats yours. Your explanation has to cover every allowed x, not just the ones you happened to test.

23. Somebody proposes a cut that gives a bigger volume than yours. Give the number they most likely picked, and explain exactly which rule it breaks.

24. Suppose the grow shelf were rebuilt 6 in wide instead of 5 in. Would your design change? Say what the new answer would be and why.

Build it and see

The model says one thing. The card stock gets a vote too.

25. Cut and fold your tray. Measure the finished inside dimensions with a ruler and multiply them out.

	Predicted	Measured
Wall height		
Longer base side		
Shorter base side		
Volume		

26. Your measured volume is probably smaller than your predicted volume. Give two physical reasons why, and say roughly how many cubic inches each one costs you.

27. Fill the tray with rice or beans, pour it into a measuring cup, and record the volume in cups. One cup is about 14.4 in^3. How close did you get?

One last thing

Hold on to the tray. If your class does the 16 in by 10 in poster board version or the 36 in by 24 in cardboard version later, comparing the three trays side by side is where the pattern in the answers shows up.

What the project is

Students cut four equal squares from the corners of an 8.5 in by 11 in sheet, fold up the flaps, and end up with an open-top tray. The volume is a cubic in the size of the corner cut. They build that function, restrict its domain, tabulate it, graph it, estimate the maximum, and then argue for one specific cut size against a written design brief.

> **The move that makes this project worth running**
>
> The brief includes a shelf the tray has to fit. That rule pushes the mathematical maximum **out of bounds**. Students who stop at "find the peak" get an answer that is genuinely wrong, and the ones who read the brief get a different number. Part G stops being a paragraph of filler and starts being an argument.

Logistics

Course	Algebra 2, or a strong Algebra 1 class after polynomial multiplication. Also works as precalculus review.
Time	Two 50-minute periods plus the build, or one period with Parts C through E set as homework.
Grouping	Pairs for Parts A through F, individual writing for Part G. The written defence is where copying shows up, so keep it individual.
Materials	Card stock or manila folders cut to 8.5 by 11, rulers with eighths, scissors, tape, and rice or dry beans if you want the volume check at the end.
Technology	Optional. Every number in the packet can be reached with a four-function calculator. Graphing software makes Part E faster but removes most of the thinking.

Standards

Standard	Where it shows up
HSA.CED.A.2	Part A, writing V(x) from the picture.
HSA.CED.A.3	Parts B and F, representing the brief as constraints and judging which solutions are viable. This is the spine of the project.
HSA.SSE.A.2	Part A, moving between factored and standard form.
HSA.APR.B.3	Part A question 5, zeros of the cubic and what each one means physically.
HSF.IF.B.4	Part D, reading maximum and end behaviour off the graph.
HSF.IF.B.5	Part B, relating the domain to the quantity it describes.
HSF.IF.C.7c	Part D, graphing a polynomial from zeros and a table.
HSG.MG.A.3	Parts F and G, applying geometry to a design problem with physical constraints.

Standard	Where it shows up
MP.3 and MP.4	Part G, building and defending a model.

The numbers, so you can grade quickly

Every value below was checked with a computer algebra system before publication.

The model

$V(x) = x(11 - 2x)(8.5 - 2x) = 4x^3 - 39x^2 + 93.5x$

Zeros at $x = 0$, $x = 4.25$ and $x = 5.5$. Only the first two are reachable on a real sheet.

Domain from the sheet alone: $0 < x < 4.25$.

Feasible interval once the brief is applied: $1.75 \le x < 4.25$.

Part C table

x (in)	Base (in)	Volume (in³)	Meets R2 and R3?
0.25	10.50 × 8.00	21.00	no
0.50	10.00 × 7.50	37.50	no
0.75	9.50 × 7.00	49.88	no
1.00	9.00 × 6.50	58.50	no
1.25	8.50 × 6.00	63.75	no
1.50	8.00 × 5.50	66.00	no
1.75	7.50 × 5.00	65.63	**yes**
2.00	7.00 × 4.50	63.00	yes
2.25	6.50 × 4.00	58.50	yes
2.50	6.00 × 3.50	52.50	yes
3.00	5.00 × 2.50	37.50	yes
3.50	4.00 × 1.50	21.00	yes
4.00	3.00 × 0.50	6.00	yes

Watch for this in question 13

$x = 1.00$ and $x = 2.25$ both give exactly 58.50 in³. One tray is 9 by 6.5 with 1 in walls, the other is 6.5 by 4 with 2.25 in walls. Same volume, very different object. It is the cleanest way into the idea that a horizontal line can cut a cubic more than once.

Answers, Part by Part

Part A

Q	Answer
1	Height x, longer base side $11 - 2x$, shorter base side $8.5 - 2x$.
2	$V(x) = x(11 - 2x)(8.5 - 2x)$.
3	$V(x) = 4x^3 - 39x^2 + 93.5x$.
4	Degree 3, leading coefficient 4. Positive leading coefficient on an odd degree, so the curve falls on the far left and rises on the far right. Neither end matters on a real sheet, which is the point of Part B.
5	$x = 0, 4.25, 5.5$. At $x = 0$ nothing is cut and there is no box. At $x = 4.25$ the short side closes to zero. $x = 5.5$ would need the short side to be cut past zero, so no sheet produces it.

Part B

Q	Answer
6	A flat sheet. No walls, no volume.
7	$8.5 - 2x > 0$, so $x < 4.25$. The short side is what binds, not the long one.
8	$0 < x < 4.25$.
9	$x \geq 1$.
10	Width fit: $8.5 - 2x \leq 5$ gives $x \geq 1.75$. Depth fit: $11 - 2x \leq 8$ gives $x \geq 1.5$. Both have to hold.
11	$1.75 \leq x < 4.25$. The width fit is the binding rule; the 1 in wall rule and the depth fit are already satisfied by everything in that interval.

Parts D and E

Q	Answer
14, 15	P is the peak of the curve at $x \approx 1.585$, $V \approx 66.15$. Q sits at the left end of the feasible interval, $x = 1.75$, $V = 65.625$. P and Q are different points, and noticing that is the whole project.
16	The peak is at $x \approx 1.59$ in with $V \approx 66.15$ in^3. Two decimal places is about as far as a 0.02 table can honestly take them. The exact value is $13/4 - \sqrt{399}/12 \approx 1.5854$.
17	$1.500 \to 66.000$, $1.625 \to 66.117$, $1.750 \to 65.625$, $1.875 \to 64.570$. The best eighth on the whole curve is 1 5/8 in.
18	Outside it. The peak sits at 1.585, and the shelf rule does not allow anything below 1.75. So the design is decided at the boundary rather than at the peak.

Parts F and G

Q	Answer
Design	$x = 1.75$ in (1 3/4 in). Base 7.5 in by 5.0 in, walls 1.75 in, volume 65.625 in^3.
19	65.625 in^3, or 65.63 to two decimals.
20	$66.148 - 65.625 = 0.523$ in^3, about 0.79 percent.
22	The argument has to cover the whole interval, not a list of tested values. The clean version: V has one turning point on $0 < x < 4.25$, at $x \approx 1.585$, and it is a maximum, so V is decreasing for every x above 1.585. The feasible interval starts at 1.75, which is already past the turn. So the largest feasible volume is at the left endpoint.
23	Almost always $x = 1.625$ (66.117 in^3) or $x = 1.5$ (66.000). Those trays are 5.25 in and 5.5 in wide. The shelf opening is 5 in. Both fail R3.
24	Yes, it changes. A 6 in opening gives $8.5 - 2x \le 6$, so $x \ge 1.25$, and the depth rule still forces $x \ge 1.5$. The peak at 1.585 is now inside the feasible interval, so the answer becomes the nearest good eighth, $x = 1.625$ in, $V = 66.117$ in^3.

Where students get stuck

Subtracting one x instead of two

$V = x(11 - x)(8.5 - x)$ is the single most common wrong model. A corner square is removed from *both* ends of every side. Send them back to the flat sheet drawing and have them trace one full edge with a finger, counting the squares they cross.

Using the long side for the domain

Plenty of students write $0 < x < 5.5$. Ask what the tray looks like at $x = 5$ and let them try to draw it.

Confusing the domain of the function with the domain of the model

The cubic itself is defined everywhere. Say the two phrases out loud as different questions, more than once.

Finding the peak and stopping

This is the failure the brief is designed to catch. Do not warn them in advance. Let Part F catch it, then let them fix it.

Rounding to a cut nobody can make

1.59 in is not a mark on a ruler. Rule R4 exists so that rounding becomes a modelling decision rather than a formatting one.

Checking one fit and forgetting the other

The shelf has a width and a depth. Both produce an inequality. Students who check only the width still land on 1.75, so they look right for the wrong reason. Ask for both.

Arguing from a table instead of from the function

"I tested every value" is false when the step is 0.25. Push for a statement about the whole interval: one turning point, so one direction of travel after it.

Reporting the built tray as a failure

The measured volume comes out under the prediction because folds have thickness and card does not crease on a line. That gap is a finding, not an error.

Running it

Two-period version

When	What
Day 1, first 10 min	Hand each pair a sheet and let them cut a tray with any corner size they like. No maths yet. Collect three trays with visibly different shapes and stand them at the front.
Day 1, 30 min	Parts A and B. Stop the class when the first pair writes $11 - x$ and put both versions on the board without saying which is right.
Day 1, last 10 min	Start Part C. Finish it for homework.
Day 2, 25 min	Parts D and E. A quick vote on the best cut before Part F is worth the two minutes it costs.
Day 2, 25 min	Part F, then Part G in silence. The moment a pair discovers 1.625 in does not fit the shelf is the moment the project works.

One-period version

Give out Part A and B completed, keep Parts C through G. That preserves the constraint argument, which is the part worth the time, and drops the algebra the class has probably already done elsewhere.

If it goes fast

Two other resources on this site now build on the same idea. The poster board packet uses a 16 in by 10 in sheet where the maximum lands on a whole number, and the extension questions that go with it include a scaling result and a proof that the best cut on a square sheet is always one sixth of the side. The advanced version uses a 36 in by 24 in sheet whose exact maximum is irrational. None of the three shares its numbers with this letter sheet packet, since each uses its own sheet size, so treat them as a further activity rather than a direct extension of this table.

A note on the build

Do the build. A class that only graphs the function will argue about the third decimal place. A class holding two trays that differ by half a cubic inch argues about whether the difference is worth measuring, which is the better argument.

How to use this

This is one worked response, written the way a student would write it. It is not the only right answer, and the wording is deliberately plain rather than polished. Hand it out after the project, not before.

Part A · Build the model

Each corner square has side x. When I fold the flaps up, the flap height is the side of the square I cut, so the walls are x inches tall.
The long side of the sheet is 11 in. I cut a square of side x off *each* end of it, so the base loses 2x and the longer base side is $11 - 2x$. The same thing happens on the 8.5 in side, so the shorter base side is $8.5 - 2x$.

Question 2 and 3

$V(x) = x(11 - 2x)(8.5 - 2x)$

Expanding: $(11 - 2x)(8.5 - 2x) = 93.5 - 22x - 17x + 4x^2 = 4x^2 - 39x + 93.5$

So $V(x) = 4x^3 - 39x^2 + 93.5x$

Check at $x = 1$: factored gives $1 \cdot 9 \cdot 6.5 = 58.5$, expanded gives $4 - 39 + 93.5 = 58.5$. They agree.

Question 4. Degree 3, leading coefficient 4. The degree is odd and the leading coefficient is positive, so far to the left the curve goes down and far to the right it goes up. Neither of those ends is a real tray, so the end behaviour is not much use here except as a check that my cubic is the right shape in the middle.

Question 5. Setting the factored form to zero gives $x = 0$, $x = 4.25$ and $x = 5.5$. At $x = 0$ I have not cut anything and there is no tray. At $x = 4.25$ the 8.5 in side has closed up completely and the base has no width. $x = 5.5$ would mean cutting 11 in of material off an 8.5 in side, which is not something a sheet of card can do. So two of the zeros are real events and one is only a root of the polynomial.

Part B · What x is allowed to be

Question 6. At $x = 0$ I would be holding the flat sheet. No walls, no volume.

Question 7. The shorter side has to survive: $8.5 - 2x > 0$, so $x < 4.25$. I checked the long side too, $11 - 2x > 0$ gives $x < 5.5$, but 4.25 is smaller so that is the one that binds. The short side always runs out first.

Question 8. Domain from the sheet: $0 < x < 4.25$.

Turning the brief into inequalities

Rule	Inequality	Solves to
R2, walls at least 1 in	$x \geq 1$	$x \geq 1$
R3, base fits a 5 in opening	$8.5 - 2x \leq 5$	**$x \geq 1.75$**
R3, base fits an 8 in depth	$11 - 2x \leq 8$	$x \geq 1.5$
R4, cuts to the nearest 1/8	x is a whole number of eighths	—

Question 11. All three of the first rules have to hold at once, so I take the strictest lower bound, which is 1.75, and keep the upper bound from the sheet. **Feasible interval: $1.75 \leq x < 4.25$.** The wall rule and the depth rule are already satisfied by everything in that interval, so the shelf width is doing all the work.

Something I noticed

The rule that ends up mattering is not the one that sounds most important. The wall height rule looks like the serious constraint when you read the brief, and it turns out to change nothing.

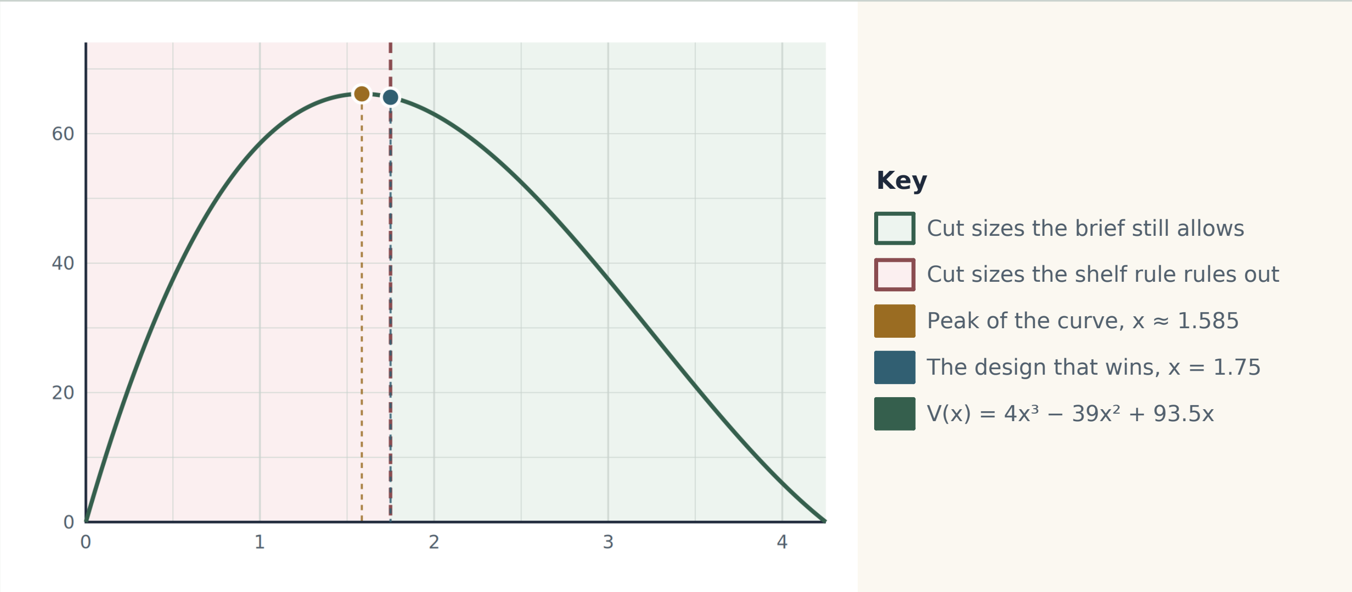
Part C · Testing cut sizes

x (in)	Base (in)	Volume (in³)	Meets R2 and R3?
0.25	10.50 × 8.00	21.00	no
0.50	10.00 × 7.50	37.50	no
0.75	9.50 × 7.00	49.88	no
1.00	9.00 × 6.50	58.50	no
1.25	8.50 × 6.00	63.75	no
1.50	8.00 × 5.50	66.00	no
1.75	7.50 × 5.00	65.63	yes
2.00	7.00 × 4.50	63.00	yes
2.25	6.50 × 4.00	58.50	yes
2.50	6.00 × 3.50	52.50	yes
3.00	5.00 × 2.50	37.50	yes
3.50	4.00 × 1.50	21.00	yes
4.00	3.00 × 0.50	6.00	yes

Question 12. The volume climbs to 1.50 and has dropped again by 1.75, so the turn happens somewhere between them.

Question 13. $x = 1.00$ and $x = 2.25$ give exactly the same volume, 58.50 in³. The first tray is 9 by 6.5 with 1 in walls, wide and shallow. The second is 6.5 by 4 with 2.25 in walls, narrow and deep. Same amount of soil, completely different tray. Once I saw that I understood why the graph has to come back down: a horizontal line can cross the curve twice.

Parts D and E · Graph and refine



My curve with the shelf rule marked. Everything to the left of the dashed line is a tray that will not go in the shelf.

Refining, step 0.10

x	1.30	1.40	1.50	1.60	1.70	1.80	1.90
V(x)	64.43	65.44	66.00	66.14	65.89	65.27	64.30

Refining, step 0.02

x	1.54	1.56	1.58	1.60	1.62	1.64
V(x)	66.107	66.135	66.148	66.144	66.125	66.089

Question 16. The peak is at about $x = 1.59$ in and the volume there is about 66.15 in³. I am confident about 1.5 and fairly confident about the second decimal, because the values at 1.58 and 1.60 differ by only 0.004. I would not claim a third decimal from this table.

Question 17. The eighths near the peak:

Cut	As a decimal	Volume (in³)	Base width (in)	Fits 5 in?
1 1/2	1.500	66.000	5.50	no
1 5/8	1.625	66.117	5.25	no
1 3/4	1.750	65.625	5.00	**yes**
1 7/8	1.875	64.570	4.75	yes

Question 18. The peak is outside the feasible interval. It sits at 1.585 and the shelf rule will not let me go below 1.75. So the design is not decided by the top of the curve at all; it is decided by the edge of what the shelf allows.

Parts F and G · The design and the defence

My design

Corner cut $x = 1\ 3/4$ in. Base 7.5 in by 5.0 in, walls 1.75 in tall, volume 65.625 in^3.

Rule	What I checked	My value	Pass
R1	Four equal squares, one sheet	1.75 in each	yes
R2	Walls at least 1 in	1.75 in	yes
R3	Base width at most 5 in	5.00 in	yes, exactly
R3	Base depth at most 8 in	7.50 in	yes
R4	A whole number of eighths	14 eighths	yes

Question 20. The best the curve can do anywhere is 66.148 in^3 at $x \approx 1.585$. My tray holds 65.625 in^3. The shelf rule costs me 0.523 in^3, which is $0.523 \div 66.148 \approx 0.79$ percent. Less than one percent, for a tray that actually goes in the shelf.

Question 22 · Why nothing allowed beats it

V has exactly one turning point on $0 < x < 4.25$, and it is a maximum, at $x \approx 1.585$. To the right of that point the curve only goes down. My feasible interval is $1.75 \le x < 4.25$, and every number in it is to the right of 1.585. So V is decreasing across the whole feasible interval, which means the largest allowed volume has to be at the smallest allowed x. That is 1.75. This covers every allowed cut, including the ones I never put in a table, because it is an argument about the shape of the curve rather than a list of tested values.

Question 23 · The answer somebody else would give

They would say 1 5/8 in, because it holds 66.117 in^3, which beats mine by half a cubic inch. It is the best eighth on the curve and it is the right answer to a different question. The base of that tray measures 7.75 in by 5.25 in, and the shelf opening is 5 in wide. It is a quarter of an inch too wide, so it breaks R3 and never gets into the shelf.

Question 24 · A 6 in shelf

With a 6 in opening the width rule becomes $8.5 - 2x \le 6$, so $x \ge 1.25$. The depth rule still gives $x \ge 1.5$, so the feasible interval becomes $1.5 \le x < 4.25$. Now the peak at 1.585 is inside it, so the constraint stops deciding the answer and the curve takes over. The nearest eighths are 1.5 and 1.625, and 1.625 wins with 66.117 in^3. So the design would change to a 1 5/8 in cut. One inch of shelf is worth about half a cubic inch of soil.

Building it

	Predicted	Measured
Wall height	1.75 in	1.69 in
Longer base side	7.50 in	7.44 in
Shorter base side	5.00 in	4.94 in
Volume	65.63 in^3	62.11 in^3

Question 26. My tray came out about 3.5 in^3 short, roughly five percent. Two reasons. First, the card has thickness, so the fold does not happen on a line, it happens over about a sixteenth of an inch, and every fold eats into the base. Four folds at a sixteenth each is most of the loss, maybe two and a half cubic inches. Second, the walls do not stand perfectly upright once they are taped; they lean in slightly at the top, which costs a bit more.

Question 27. The rice filled the tray to 4.3 cups. At 14.4 in^3 per cup that is about 61.9 in^3, which is within a quarter of a cubic inch of what I measured with the ruler. So the two physical measurements agree with each other, and both sit about five percent under the model. The model is not wrong; it just describes a tray made of something with no thickness.

What I would tell the garden club

Cut 1 3/4 in squares. The tray holds 65.6 in^3 on paper and about 62 in practice, and it slides into the shelf with nothing to spare. A 1 5/8 in cut holds more and does not fit, which makes it worth nothing at all.