Open-Top Box Design Challenge

Advanced version · 36 in by 24 in sheet · answers included

Burke Tutoring in Fremont
Polynomial modeling project

When to use this

The same construction on a much larger sheet. The maximum is irrational, so there is no tidy answer to land on, and question 4 asks students to divide a cubic by a linear factor they have to find first.

I use this with students who finished the shorter versions quickly and were not stretched by them. It suits an honours group, or a precalculus class revisiting polynomials before optimisation.

The problem

A flat sheet of corrugated card measures 36 in by 24 in. Cut four equal squares from the corners, fold the sides up, and tape them. The construction is the same as the poster board version; the numbers are not.

1. Write V(x) in factored and standard form, and state the domain.

2. Find the exact value of x that maximises the volume. Leave your answer as a radical, then give it to three decimal places. Both critical values come out of the same quadratic, so explain how you know which one to discard.

3. Find the exact maximum volume, in the form $a + b\sqrt{c}$.

4. Find **every** corner cut that gives a volume of exactly 1620 in^3. Test $x = 3$ first, divide the cubic by the factor that gives you, then solve what is left. How many of your answers are physically possible?

5. The finished tray has to fit a pallet slot 15 in wide. Write the inequality, find the feasible interval, and say whether the unconstrained maximum survives it. Compare what happens here with what happened on the 16 in by 10 in sheet.

6. A supplier offers you 864 in² of card in any rectangle you like, and 36 by 24 is one option. Investigate at least three others and say which shape you would order.

Answers

1. $V(x) = x(36 - 2x)(24 - 2x) = 4x^3 - 120x^2 + 864x$, on the domain $0 < x < 12$.

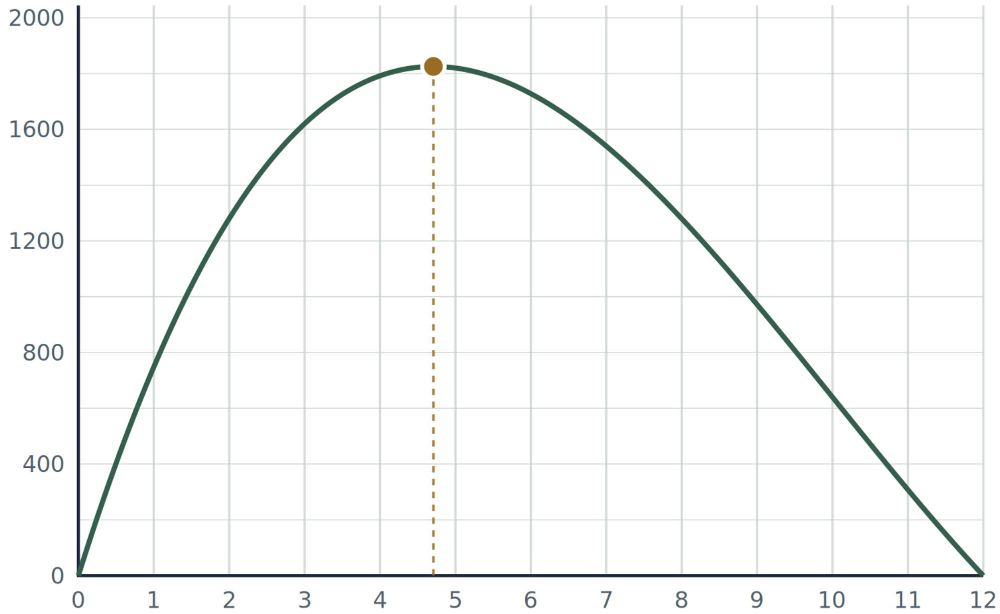
2. $V'(x) = 12x^2 - 240x + 864 = 12(x^2 - 20x + 72)$, so $x = 10 \pm 2\sqrt{7}$. The root $10 + 2\sqrt{7} \approx 15.29$ lies outside $0 < x < 12$, so the maximum is at $x = 10 - 2\sqrt{7} \approx 4.708$ in. Students who test a value either side, and students who notice the cubic must return to zero at $x = 12$, both reach this.

3. $V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825.297$ in³.

4. $V(3) = 3 \cdot 30 \cdot 18 = 1620$, so $(x - 3)$ is a factor. Dividing $4x^3 - 120x^2 + 864x - 1620$ by 4 and then by $(x - 3)$ leaves $x^2 - 27x + 135$, giving $x = (27 \pm 3\sqrt{21})/2$, that is 6.626 and 20.374. Three real solutions, two of them physical: at $x = 20.374$, twice that is more than the 24 in side, so the two cuts on that side would overlap and no real sheet can produce it.

5. $24 - 2x \le 15$ gives $x \ge 4.5$, so the feasible interval is $4.5 \le x < 12$. The maximum at 4.708 survives, because it is already above 4.5. This is the opposite of what happens on the 16 by 10 sheet, where the same kind of rule pushes the maximum out of range, and asking students to account for the difference is worth more than either result on its own.

6. Rectangles of area 864 in²: 48 by 18 gives 1600 in³, 36 by 24 gives 1825, 32 by 27 gives 1871, and the square of side about 29.39 in gives 1881. The closer to square, the larger the tray, which matches the 160 in² investigation on the extension sheet.



The volume curve for the 36 in by 24 in sheet on $0 < x < 12$, with the maximum marked in gold.